On Certain Problems in the Theory of Normed-Spaces

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C E R T I F I C A T E

This is to certify that the contents of this thesis, entitled, " ON CERTAIN PROBLEMS IN THE THEORY OF NORMED SPACES " is the original research work of Mr. Ataullah Siddiqi carried out under my supervision.

I further certify that the work has not been submitted either partly or fully to any other University or Institution for the award of any degree.

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CONTENTS

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Ataullah Siddiqi

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P R E F A C E

The present thesis entitled " On Certain Problems in the Theory of Normed Spaces " is outcome of my researches since Sept.1975, under the able guidance of Prof.A.H.Siddiqi. Several classes of normed spaces like non-archimedean normed space, p-normed space or quasi-normed space, normed lattices, 2-normed space and n-normed space etc, have attracted the attention of several eminent mathematicians in the last 50 years. A person working in the domain of normed spaces is delighted to know that only mathematician L.V. Kanterovich to share the Noble prize of Economics in the archited of normed lattices. Inspired by these works we have studied in the present thesis, some new classes of normed spaces like, quasi-2-normed spaces, quasi-2-normed lattices, non-archimedean quasi-2-normed spaces and non-archimedean n-normed spaces besides several related concepts.

The present thesis consists of 7 chapters. In the first chapter, we give a résumé of hitherto known results which have inter-connections with our investigations. Chapter II is concerned with the study of quasi-2-normed spaces which was introduced by A. H. Siddiqi and Rafhanizadeh in 1974. Chapter III deals with the quasi-2-normed lattices which Chapter IV is devoted to non-archimedean quasi-2-normed spaces. In Chapter V, we have studied ultra-n-metric space which is n-dimensional analogue of ultra-metric space. Chapter VI contains results concerning n-dimensional analogue of normed space over valued fields namely n-normed space over valued fields. In Chapter VII a beautiful theorem is proved, which contains a well known theorem of A.F.Monna concerning the characterisation of convergence preserving matrices in non-archimedean analysis. Towards the end we give a fairly extensive bibliography of various publications which have been referred in the present thesis.

It may be mentioned that results of Chapter VI and VII have been presented in the symposium on Topology held at the Delhi University under the Chairmanship of Prof. M.K.Singhal in the last week of Dec. 1978 and symposium on Topology in the Annual Conference of Indian Science Congress held at Hyderabad in Jan.1979.

The theorem proved in Chapter VII has been very much appreciated by Prof. H. De-Grande-De-Kimple of Brussel University Belgium, a well known functional Analysist, who was on a visiting assignment in our department for a period of 3 weeks.

I would also like to mention that I have published 3 research papers in collaboration with Dr. Ahmad Khan [(i) Orthogonality in 2-normed spaces, Aligarh Bull.Math.,5-6(1975-76), 63-74(ii) characterization of strictly convex 2-normed spaces in terms of duality 2-mappings, Math. Japan 23(1978),133-137 and (iii) B-orthogonality in 2-normed spaces, accepted for publication Bull.Calcutta Math.Soc.]. But these results are not included in my thesis as that work is contained in the Ph.D. thesis of Dr. Ahmad Khan submitted in Dec. 1978.

Ataullah Siddiqi
(ATAULLAH SIDDIQI)

CHAPTER – I

INTRODUCTION

In the present chapter, we mention some well-known results from the theory of 2-normed spaces and non-archimedean functional analysis which are relevent to our investigations presented in the subsequent chapters.

1.1. 2-METRIC SPACES

Definition 1.1.1. [14]. Let R be a non-empty set and σ a function on $R \times R \times R$, with the following properties.

1. If atleast two of the points a,b,c are equal, then $\sigma(a,b,c) = o$. For two distinct points a and b of R, there is a point c in R such that $\sigma(a,b,c) \neq o$.

2. $\sigma(a,b,c) = \sigma(a,c,b) = \sigma(b,c,a)$.

3. $\sigma(a,b,c) \leq \sigma(a,b,d) + \sigma(a,d,c) + \sigma(d,b,c)$,

σ is called a 2-metric on R and (R,σ) 2-metric space.

The concept of 2-metric arises as generalisation of the area of a triangle in a Euclidean space of dimension $m \geq 2$, by choosing the 2-metric as a function defined over a set. Thus on each Euclidean space of dimension $m \geq 2$, a

3-point function

$$\sigma(a,b,c) = \frac{1}{2}\left\{\sum_{i<j}\begin{vmatrix}\alpha_i & \alpha_j & 1\\ \beta_i & \beta_j & 1\\ \gamma_i & \gamma_j & 1\end{vmatrix}^2\right\}^{1/2},$$

where $a = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $b = (\beta_1, \beta_2, \ldots, \beta_m)$ and $c = (\gamma_1, \gamma_2, \ldots, \gamma_m)$, is a 2-metric.

Let (R,σ) be a 2-metric space. We denote by $U_\alpha(a,b) = \{c \in R : \sigma(a,b,c) < \alpha\}$, for any real number $\alpha > 0$, the α-neighbourhood of a and b in R. For any $\Sigma = \{(a_1,\alpha_1), \ldots, (a_n,\alpha_n)\}$, ($a_i \in R$, α_i real and > 0) and any $a \in R$, let $W_\Sigma(a) = \bigcap_{i=1}^{n} U_{\alpha_i}(a, a_i)$ and $\overline{W}_\Sigma(a) = \bigcap_{i=1}^{n}\{c \in R :$ $(a, a_i, c) \leq \alpha_i\}$. The family of all sets $U_\alpha(a,b)$ is sub-base for a topology $\mathcal{J}$ on R generated by the 2-metric or natural topology of (R,σ). For every point a of the 2-metric space (R,σ), the family of all the sets $W_{\Sigma(a)}$ forms an open base [See Theorem 1, 14].

<u>Definition</u> 1.1.2 [14]. A 2-metric space (R,σ) is said to have the property K if the following holds :

If for sequence of points a, a_1, a_2,... of R there exists two points b and c of R with $\sigma(a,b,c) \neq 0$,

$\lim_{i \to \infty} \sigma(a,b,a_i) = 0$ and $\lim_{i \to \infty} \sigma(a,c,a_i) = 0$, then

$\lim_{i \to \infty} \sigma(a,a',a_i) = 0$ for every point a' of R.

<u>Definition</u> 1.1.3. [14]. (R,σ) has the property S if for any points a and b of R and every $\epsilon > 0$ there exists neighbourhood U_a of a and U_b of b such that for any point $a' \in U_a$ and $b' \in U_b$, $\sigma(a,a',b') < \epsilon$ holds.

Before closing the section, we mention some typical results.

<u>Theorem</u> 1.1.1 [14]. The 2-metric function σ is continuous function of three variables if, and only if it satisfies the property S.

<u>Theorem</u> 1.1.2. [14]. Let (R,σ) be a 2-metric space. Then there exists a metric d on R and a real number λ such that

$$\sigma(a,b,c) \leq \lambda \min\left\{d(a,b)d(b,c), d(b,c)d(c,a), d(c,a)d(a,b)\right\}.$$

<u>Theorem</u> 1.1.3. [14]. Every 2-metric space is regular.

Theorem 1.1.4. [14]. Let $R_1, R_2, R_3, \ldots$ be finite or

countable numbers of 2-metric spaces with 2-metrics $\sigma_1, \sigma_2, \sigma_3, \ldots,$ respectively and $R = (R_1 \times R_2 \times R_3 \times \ldots)$. Then (R, σ) is a 2-metric space where $\sigma(a,b,c) = \sum_i \sigma_i(a_i, b_i, c_i)^n$ (In case R_i are countable, $\sum_i \sigma_i(a_i, b_i, c_i)$ is assumed to be finite). The topology induced by 2-metric σ is equivalent to the product topology of R. R has property S if, and only if each R_i has property S. Similarly R has property K if, and only if each R_i has property K.

1.2. 2-NORMED SPACES

Definition 1.2.1 |15|. Let L be a linear space of dimension > 1 and $\nu(.,.)$ be real valued function on $L \times L$ satisfying the following conditions.

1. $\nu(a,b) = 0$ if, and only if a and b are linearly dependent.

2. $\nu(a,b) = \nu(b,a)$.

3. $\nu(a, \beta b) = |\beta| \, \nu(a,b)$, β is scalar.

4. $\nu(a, b+c) \leq \nu(a,b) + \nu(a,c)$.

$\nu(.,.)$ is called 2-norm on L and $(L, \nu(.,.))$ 2-normed space.

EXAMPLES OF 2-NORMED SPACES

1. Let R^3 be Euclidean vector space of dimension 3 and $a = x\vec{i} + y\vec{j} + z\vec{k}$, $b = l\vec{i} + m\vec{j} + n\vec{k}$ belong to R^3. Then $\nu(a,b) = [(yn-zm)^2 + (zl-xn)^2 + (xm-yl)^2]^{1/2}$ is a 2-norm and $(R^3, \nu(.,.))$ is a 2-normed space.

2. Let $(L, \nu(.))$ be a normed space of dimension > 1 and X be the set of all bounded linear functionals on L with norm less than or equal to 1. Then

$$\nu(a,b) = \frac{1}{2} \sup_{f,g \in X} \begin{vmatrix} f(a) & g(a) \\ f(b) & g(b) \end{vmatrix},$$

is a 2-norm on L and $(L, \nu(.,.))$ is 2-normed space.

From conditions (3) and (4) of Definition 1.2.1, it follows that 2-norm is a non-negative function.

The following theorem gives the relation between 2-normed space and 2-metric space.

Theorem 1.2.1 [15]. Every 2-normed space $(L, \nu(.,.))$ is a translation invariant 2-metric space with respect to a 2-metric σ given by

$$\sigma(a,b,c) = \nu(b-a,\ c-a).$$

Now we shall state a theorem which gives the topological structure of 2-normed space.

<u>Theorem</u> 1.2.2 [15]. Every 2-normed space L is a locally convex topological space.

<u>REMARKS</u>

1. Every 2-normed space is uniformable as topological vector space and consequently completely regular.

2. The topologies induced by a norm and 2-norm on a finite dimensional vector space are equivalent.

3. For each norm $\nu(.)$ and 2-norm $\nu(.,.)$ defined on a finite dimensional vector space L, there always exists a real positive number λ such that $\nu(a,b) \leq \lambda\nu(a)\ \nu(b)$ for all $a, b \in L$.

4. There exists a 2-normed space of uncountable dimension which is not metrizable and does not have any norm [See p.13,15].

5. Each finite dimensional 2-normed space has property K.

The function $\sigma(a,b,c) = \nu(b-a,\ c-a)$, where $\nu(.,.)$ is a 2-norm, is not a continuous function of 3 variables in general, but the following theorem gives a criterion for its continuity.

Theorem 1.2.3 [15]. A 2-metric function σ associated with 2-norm is a continuous function of all the three variables if the following condition holds. For $\epsilon > o$ there exists a neighbourhood U of a such that for all $a^*, b^* \in U$, $\nu(a^*,b^*) < \epsilon$.

The following theorem is 2-dimensional analogue of Riesz-lemma which has been proved in [24].

Theorem 1.2.4. $(L, \nu(.,.))$ be 2-normed space and M be a closed proper subspace of L. Further let $o < \epsilon < 1$ and b be a fixed point of M. Then there exists an element $a_o \in L - M$ such that $\nu(a_o,b) = 1$ and $\nu(a-a_o,b) \leq \epsilon$ for all $a \in M$.

It has been proved by Gähler, Siddiqi and Gupta [24] that all 2-norms on finite dimensional vector spaces are equivalent.

1.3. COMPLETENESS OF 2-NORMED SPACES

There are two types of completeness, one due to White [55] and other due to Gähler [20]. White calls a sequence $\{a_n\}$ a Cauchy sequence in 2-normed space $(L, \nu(.,.))$ if there exists points $a, b \in L$ with $\nu(a,b) \neq o$ such that $\lim_{n,m \to \infty} \nu(a_n-a_m,a) = \lim_{n,m \to \infty} \nu(a_n-a_m,b) = o$. A sequence

$\{a_n\}$ is convergent sequence in L if there exists a point $a \in L$ such that $\lim\limits_{n \to \infty} \nu(a_n - a, b) = o$ for all $b \in L$. A 2-normed space is complete 2-normed space or 2- Banach space in the sense of White if every Cauchy sequence is convergent in his sense. If $(L, \nu(.,.))$ is a 2-normed space, for any $b \in L$, $\mu_b(a) = \nu(a,b)$ is semi-norm. A Moor-smith sequence $\{b_i\}$, $b_i \in L$, is called Cauchy if it is Cauchy with respect to all semi-norms μ_b, $b \in L$ [20]. A 2-normed space is called complete 2-normed space or 2-Banach space in the sense of Gähler [20] if every Cauchy Moor-smith sequence is convergent with respect to semi-norm μ_b. A 2-normed space is called strong locally bounded if for $a, b \in L$ with $\nu(a,b) \neq o$ and each $\varepsilon > o$ the set $W_{\Sigma(o)}$, $\Sigma = \{(a,\varepsilon),(b,\varepsilon)\}$ is bounded. Every strong locally bounded 2-normed space is locally bounded, but every locally bounded 2-normed space is not strong locally bounded. Every countable Cauchy sequence in the sense of Gähler is Cauchy sequence in the sense of White but converse is not true. Each strong locally bounded 2-normed space is 2-Banach space in the sense of Gähler and consequently in the sense of White. Conversely each 2-Banach space in the sense of White is strong locally bounded and so it is 2-Banach space in the sense of Gähler. Each finite dimensional 2-normed space is 2-Banach space. It is proved in the interesting way [20] that L is a 2-Banach space whenever it is a Banach space.

1.4. 2-INNER PRODUCT SPACES (2-PRE-HILBERT SPACES)

Definition 1.4.1 [7]. Suppose L is a vector space of dimension greater than 1 and $(.,./.)$ is a real function defined on $L \times L \times L$ which satisfies the following conditions.

1. $(a,a/b) \geq o$, $(a,a/b) = o$ if, and only if a and b are linearly dependent.

2. $(a,a/b) = (b,b/a)$.

3. $(a,b/c) = (b,a/c)$.

4. $(\alpha a,b/c) = \alpha(a,b/c)$, α is scalar.

5. $(a+a',b/c) = (a,b/c) + (a',b/c)$.

$(.,./.)$ is called 2-inner product and $(L,(.,./.))$ a 2-inner product space or 2-pre-Hilbert space.

Examples [7].

1. Let l_2 be the space of sequences $\{a_i\}$ such that $\sum_{i=1}^{\infty} |a_i|^2 < \infty$. Then $(l_2,(.,./.))$ is 2-pre-Hilbert space with 2-inner product defined as follows.

$$(a,b/c) = \sum_{i<j} (\alpha_i\gamma_j - \gamma_i\alpha_j)(\beta_i\gamma_j - \beta_j\gamma_i) ,$$

where $a = (\alpha_1,\alpha_2,\ldots,\alpha_n)$, $b = (\beta_1,\beta_2,\ldots,\beta_n)$, $c = (\gamma_1,\gamma_2,\ldots,\gamma_n)$.

Let $(L,(.,.))$ be a pre-Hilbert space. Then $(L,(.,./.))$ is a 2-inner product space with

$$(a,b/c) = (a,b) \quad - (a,c)\ (b,c).$$

The following results are 2-dimensional analogue of Cauchy Buniakowski inequality and parallelogram law.

<u>Theorem</u> 1.4.1 [7]. If L is 2-pre-Hilbert space, then

$$(a,b/c) \leq (a,a/c)^{1/2} (b,b/c)^{1/2} \text{ for all } a,b,c \in L.$$

<u>Theorem</u> 1.4.2 [7]. On any 2-pre-Hilbert space $(L,(.,./.))$

$\nu(a,b) = \sqrt{(a,a/b)}$ defines a 2-norm for which

$$(a,b/c) = \frac{1}{4}\left(\nu(a+b,c)^2 - \nu(a-b,c)^2\right) \text{ and}$$

$$\nu(a+b,c)^2 + \nu(a-b,c)^2 = 2(\nu(a,c)^2 + \nu(b,c)^2).$$

1.5. <u>NON-ARCHIMEDEAN FUNCTIONAL ANALYSIS</u>

<u>Definition</u> 1.5.1 [34]. Let X be a non-empty set. A class $\mathcal{J}$ of subsets of X is called a topology on X if it satisfies the following conditions.

(i) The empty set ϕ and X belong to $\mathcal{J}$.

(ii) The union of every class of sets in $\mathcal{J}$ is a set in $\mathcal{J}$.

(iii) The finite intersection of every finite class of sets in $\mathcal{J}$ is a set in $\mathcal{J}$.

The pair $(X,\mathcal{J})$ is called a topological space. The numbers of $\mathcal{J}$ are called the open sets and the complement of an open set is said to be closed.

A topological space which is also a vector space such that the mapping $(x,y) \to x+y$ and $(\alpha,x) \to \alpha x$ are continuous with respect to the topology is called Topological vector space.

__Definition__ 1.5.2 [34]. A topological space is said to be locally compact if every point of it has a neighbourhood with compact closure.

__Definition__ 1.5.3 [34]. A topological vector space E over a field F is called locally convex if the fundamental system of neighbourhoods of zero consists of only convex sets.

__Definition__ 1.5.4 [44a]. A subset A of a topological vector space over field K is called K-convex if $\alpha,\beta \in K$ and $x,y \in A$ such that $|\alpha| \leq 1$, $|\beta| \leq 1$ implies $\alpha x+\beta y \in A$.

Definition 1.5.5 [44]. Let K be a field with complete non-archimedean valuation. A topological space E over the field K is called locally K-convex if the fundamental system of neighbourhoods of zero consists of only K-convex sets.

Definition 1.5.6 [44]. A metric space is called ultra metric space if it satisfies the stronger condition,

$$\rho(x,y) \leq \max\{ \rho(x,z), \rho(z,y)\}.$$

Definition 1.5.7 [23]. A 2-metric space is called ultra 2-metric space if it satisfies the stronger condition

$$\sigma(a,b,c) \leq \max\{\sigma(a,b,d), \sigma(a,d,c), \sigma(d,b,c).$$

Definition 1.5.8 [44]. If a norm function $\nu(.)$ satisfies the stronger relation $\nu(x+y) \leq \max\{\nu(x), \nu(y)\}$ then it is called non-archimedean norm and a vector space together with non-archimedean norm is called non-archimedean normed space. If it is complete with respect to metric induced by non-archimedean norm then we call it non-archimedean Banach space.

Definition 1.5.9 [23]. A 2-norm $\nu(.,.)$ over valued field K is called non-archimedean 2-norm of it satisfies the following condition :

$$\nu(a,b) \leq \max\left\{\nu(a,c),\ \nu(c,b)\right\}.$$

The vector space X over valued field K together with non-archimedean 2-norm, $\nu(.,.)$ is called non-archimedean 2-normed space. If the 2-normed space X is complete with respect to semi-norm induced by non-archimedean 2-norm then it is called 2-Banach space.

For further details of non-archimedean functional analysis kindly see [44]; [44a] and [48].

CHAPTER - II

CERTAIN PROBLEMS IN QUASI-2-NORMED SPACES

Esfahanizadeh and Siddiqi [12,13], have investigated some fundamental properties of quasi-2-normed spaces. In the present chapter, we obtain some more results in this direction. There are three sections in this chapter. Section 2.1, deals with the Bivectors and quasi-2-normed spaces while in Section 2.2, two theorems concerning Strict convexity are mentioned. Section 2.3, is devoted to the elementary properties of sequences in 2-normed spaces and 2-functionals.

2.1. BIVECTORS AND QUASI-2-NORMED SPACES

Definition 2.1.1 [12]. Let L be a linear space of dimension greater than 1 and $\nu(.,.)$ a real-valued function over $L \times L$ which satisfies the following four conditions ;

1. $\nu(a,b) = o$ if, and only if a and b are linearly dependent,

2. $\nu(a,b) = \nu(b,a)$,

3. $\nu(a,\beta b) = |\beta|^p \nu(a,b)$, when β is real and $o < p \leq 1$,

4. $\nu(a,b+c) \leq \nu(a,b) + \nu(a,c)$.

$\nu(.,.)$ is called a quasi-2-norm with power p on L and $(L, \nu(.,.))$ quasi-2-normed space with power p.

If L is a linear space of dimension greater than 1, let $B_L^!$ be the set of all formal expressions $\sum_{i=1}^{m} a_i \times b_i$, where a_i, b_i, ($i = 1, \ldots, m$) are vectors in L. Let $\sim$ be the equivalence relation on $B_L^!$ defined by

$$\sum_{i=1}^{m} a_i \times b_i \sim \sum_{i=1}^{m'} a_i' \times b_i'$$

if, and only if for arbitrary linear functionals f and g on L,

$$\sum_{i=1}^{m} \begin{vmatrix} f(a_i) & g(a_i) \\ f(b_i) & g(b_i) \end{vmatrix} = \sum_{i=1}^{m'} \begin{vmatrix} f(a_i') & g(a_i') \\ f(b_i') & g(b_i') \end{vmatrix} .$$

Let B_L be the quotient space $B_L^! / \sim$. The elements of B_L are called bivectors over L([15,20]) and the elements of $B_L^!$ belonging to a bivector are called representatives of this bivector. The bivector with the representative $\sum_{i=1}^{m} a_i \times b_i$ will be denoted by $\mathfrak{b}(\sum_{i=1}^{m} a_i \times b_i)$. If a bivector has a representative of the form $\sum_{i=1}^{1} a_i \times b_i = a_1 \times b_1$, then it is said to be simple. Only in the case where L has dimension less than or equal to 3, every bivector over L turn out to be simple [15]. The space B_L is a linear space with

$$\sigma\left(\sum_{i=1}^{m} a_i \times b_i\right) + \sigma\left(\sum_{i=1}^{n} a_{i+m} \times b_{i+m}\right) = \sigma\left(\sum_{i=1}^{m+n} a_i \times b_i\right)$$

and

$$\beta\,\sigma\left(\sum_{i=1}^{m} a_i \times b_i\right) = \sigma\left(\sum_{i=1}^{m} a_i \times \beta b_i\right), \text{ where } \beta \text{ is real.}$$

If $\nu(.)$ is a norm on B_L, then $\nu(a,b) = \nu(\sigma(a \times b))$ defines a 2-norm on L [15]. There is an example in [4] which shows that for every 2-norm $\nu(.,.)$ on L, there need not exist a norm $\nu(.)$ on B_L, which satisfies $\nu(\sigma(a \times b)) = \nu(a,b)$ for all $a,b \in L$.

__Theorem__ 2.1.1. For every quasi-normed space L, the function

$$\nu_p(b\,(a \times b)) = \tfrac{1}{2}\left(\sup\left\{\,|f(a)g(b) - g(a)f(b)|^{p}\right\}\right),$$

$$0 < p \leq 1;\ \nu(f),\ \nu(g) \leq 1$$

over B_L, forms a quasi-norm.

__Proof__. Since for an arbitrary representation $a \times b$ of σ and arbitrary linear functional f and g

$$\frac{1}{2}\ |f(a)g(b)-g(a)f(b)|^{p} \leq \frac{1}{2}\left\{|f(a)|^{p}|g(b)|^{p}+|g(a)|^{p}|f(b)|^{p}\right\}$$

$$\leq \frac{1}{2}\left\{2\ \nu_p(f)\ \nu_p(g)\,\nu_p(a)\ \nu_p(b)\right\}$$

$$\leq \nu_p(a)\ \nu_p(b),\ \text{when}\ \nu(f),\nu(g)\leq 1$$

holds, $_p(b\ )$ is finite.

Now, we shall prove that $\nu_p(\mathfrak{b}\ ) = o$ if, and only if $\mathfrak{b}$ is null bivector. Proceeding on the lines of the proof of Theorem 14 [15], we can easily prove that $\nu_p(\mathfrak{b}\ ) = o$ implies that $\frac{1}{2}$ sup $\{|f(a)g(b)-g(a)f(b)|^p\}=o$, which implies that $\mathfrak{b}$ is null bivector and conversely. Since $\beta\mathfrak{b}\ (a\times b) = \mathfrak{b}\ (a\ \times\beta b)$, we have

$$\nu_p(\beta b\ ) = \frac{1}{2}\ \sup\left\{|f(a)g(\beta b)-g(a)f(\beta b)|^{p}\right\}$$

$$= \frac{1}{2}\ \sup\ \left\{|\beta(f(a)g(b))-\ \beta(g(a)f(b))|^{p}\right\}$$

$$= \frac{1}{2}\ \sup\ \left\{|\beta(f(a)g(b)-\ g(a)f(b))|^{p}\right\}$$

$$= \frac{1}{2}\ \sup\quad |\beta|^{p}\ \left\{|f(a)g(b)-g(a)f(b)\ \ |^{p}\right\}$$

$$=\ \ |\beta|^{p}\ \frac{1}{2}\ \sup\ \left\{|f(a)g(b)-g(a)f(b)|^{p}\right\}$$

$$= |\beta|^{p}\ \ \nu_p(\mathfrak{b}\ ).$$

Now, we shall show that

$$\nu_p(\acute{b}_1+\acute{b}_2) \leq \nu_p(\acute{b}_1)+ \nu_p(\acute{b}_2),$$

where $\acute{b}_1 = \acute{b}(a \times b)$ and $\acute{b}_2 = \acute{b}(c \times d)$.

We have

$$\nu_p(\acute{b}_1+\acute{b}_2) = \tfrac{1}{2} \sup \{ |f(a)g(b)-g(a)f(b)+ +f(c)g(d)- g(c)f(d)|^p \}$$

$$\leq \tfrac{1}{2} \sup |f(a)g(b)-g(a)f(b)|^p + + \tfrac{1}{2} \sup |f(c)g(d)-g(c)f(d)|^p$$

$$= \nu_p(\acute{b}_1) + \nu_p(\acute{b}_2).$$

Therefore

$$\nu_p (\acute{b}_1 + \acute{b}_2) \leq \nu_p(\acute{b}_1)+ \nu_p(\acute{b}_2).$$

This completes the proof of the theorem.

<u>Theorem</u> 2.1.2. Let L be a linear space of dimension greater than 1. From every quasi-norm $\nu_p(\acute{b})$ over B_L, we can obtain a quasi-2-norm $\nu(a,b)$ with power p, $0<p\leq 1$ over L by the relation

$$\nu(a,b) = \nu_p(\sigma(a \times b))$$

Proof.

1. $\nu(a,b) = \nu_p(\sigma(a \times b)) = o$ if the vectors a and b are linearly dependent.

2. $\nu(a,b) = \nu_p(\sigma(a \times b)) = \nu_p(\sigma(b \times a)) = \nu(b,a)$.

3. $\nu(a,\beta b) = \nu_p(\sigma(a \times \beta b)) = |\beta|^p \nu_p(\sigma(a \times b)) =$

$$= |\beta|^p \nu(a,b),$$

$o < p \leq 1$ and for every real number β.

4. $\nu(a,b+c) = \nu_p(\sigma(a \times (b+c))) \leq \nu_p(\sigma(a \times b))$

$$+ \nu_p(\sigma(a \times c))$$

$$= \nu(a,b) + \nu(a,c).$$

Hence, $\nu(a,b)$ represents a quasi-2-norm over L with power p, $o < p \leq 1$. Combining Theorem 2.1.1 and Theorem 2.1.2, we obtain the following interesting result.

Theorem 2.1.3. For every quasi-normed space L, the function

$$\nu(a,b) = \frac{1}{2} \sup_{f,g \in F_L} |f(a)g(b) - g(a)f(b)|^p, o < p \leq 1$$

define a quasi-2-norm with power p on L.

2.2. STRICTLY CONVEX QUASI-2-NORMED SPACES

This concept has been studied by Esfahanijadeh, J. and A. H. Siddiqi [13]. In this section, we shall prove some more results concerning strictly convex.

Theorem 2.2.1. Let L be a linear space of dimension greater than 1, $\nu_p(.)$ be a quasi-norm with power p on B_L and $\nu(.,.)$ be a quasi-2-norm on L with $\nu_p(\sigma(a \times b)) = \nu_p(a,b)$, and all $a,b \in L$. If $(B_L, \nu_p(.))$ is strictly convex, then $(L, \nu_p(.,.))$ is strictly convex. If the dimension of L is less than or equal to 3 and $(L, \nu_p(.,.))$ is strictly convex, then $(L, \nu_p(.))$ is strictly convex.

Theorem 2.2.2. Let $(L, \nu_p(.,.))$ be a quasi-2-normed space and $(L', \nu_p(.))$ be a quasi-normed space.

1. If $(L, \nu_p(.,.))$ is strictly convex, C is a fixed non-zero element of L, and f is a function from L' into L which satisfies

$$\nu_p(f(a)-f(b),c) = \nu_p(a-b), \text{for every } a,b \in L'.$$

then the function g_c from L' into L_c, defined by $g_c(a) = [f(a)-f(o)]_c$, is linear.

2. If $(L', \nu_p(.))$ is strictly convex, c is a fixed non-zero element of L, and f is a function from L into L' satisfying

$$\nu_p(f(a)-f(b)) = \nu_p(a-b,c), \text{ for every } a,b \in L,$$

then the function g from L into L', defined by $g(a)=f(a)-f(o)$ is linear.

The proofs of Theorem 2.2.1 and Theorem 2.2.2 run parallel to the proofs of Theorem 2 and Theorem 3 [6], respectively.

2.3. SEQUENCES AND 2-FUNCTIONALS

In this section, we have studied some elementry properties of sequences in 2-normed spaces and 2-functionals.

Definition 2.3.1.[12]. A sequence $\{x_n\}$ in a quasi-2-normed space X is called Cauchy sequence if there exists $y, z \in X$ such that y and z are linearly independent with $\lim \nu(x_n - x_m, y) = o$ and $\lim \nu(x_n - x_m, z) = o$.

Definition 2.3.2.[12] A sequence $\{x_n\}$ in a quasi-2-normed space X is called a convergent sequence if there is an $x \in X$ such that the $\lim \nu(x_n - x, y) = o$ for all $y \in X$. If $\{x_n\}$ converges to x we write $x_n \longrightarrow x$, and X is called complete if every Cauchy sequence is convergent in it.

Theorem 2.3.1. Let X be a quasi-2-normed space.

(i) If $\{x_n\}$ is a Cauchy sequence in X with respect to a and b, then $\nu(x_n, a)$ and $\nu(x_n, b)$ are real Cauchy sequences.

(ii) If $\{x_n\}$ and $\{y_n\}$ are a Cauchy sequences in X with respect to a and b and $\{\alpha_n\}$ is a real Cauchy sequence then (a) $\{x_n + y_n\}$ and (b) $\{\alpha_n x_n\}$ are Cauchy sequences in X.

(iii) If $x_n \longrightarrow x$ and $y_n \longrightarrow y$, then $x_n + y_n \longrightarrow x+y$.

(iv) If $x_n \longrightarrow x$ and $\alpha_n \longrightarrow \alpha$ then $\alpha_n x_n \longrightarrow \alpha x$.

(v) If $\dim X \geq 2$, $x_n \longrightarrow x$ and $x_n \longrightarrow y$, then $x = y$.

We prove here only (ii(b)) and (iv).

Proof: ii(b) : In order to prove that $\{\alpha_n x_n\}$ is a Cauchy sequence we have to show that

$$\nu(\alpha_n x_n - \alpha_m x_m, a) \longrightarrow 0 \quad \text{as } n,m \longrightarrow \infty$$

and

$$\nu(\alpha_n x_n - \alpha_m x_m, b) \longrightarrow 0 \quad \text{as } n,m \longrightarrow \infty$$

$$\nu(\alpha_n x_n - \alpha_m x_m, a) = \nu(\alpha_n x_n - \alpha_n x_m + \alpha_n x_m - \alpha_m x_m, a)$$

$$\leq \nu(\alpha_n x_n - \alpha_n x_m, a) + \nu(\alpha_n x_m - \alpha_m x_m, a)$$

$$= |\alpha_n|^p \nu(x_n - x_m, a) + |\alpha_n - \alpha_m|^p \nu(x_m, a)$$

Since $\{\alpha_n\}$ and $\{\nu(x_n, a)\}$ are Cauchy sequences, the right hand side tends to zero as $n,m \longrightarrow \infty$ which proves that $\{\alpha_n x_n\}$ is a Cauchy sequence.

Proof of (iv). We have

$$\nu(\alpha_n x_n - \alpha_n, a) = \nu(\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x, a)$$

$$\leq \nu(\alpha_n x_n - \alpha_n x, a) + \nu(\alpha_n x - \alpha x, a)$$

$$= |\alpha_n|^p \nu(x_n - x, a) + |\alpha_n - \alpha|^p \nu(x, a).$$

Since $|\alpha_n|^p < \infty$, $\nu(x_n - x, a)$ and $|\alpha_n - c|^p$ tend to o as $n \longrightarrow \infty$ we see that $\alpha_n x_n \longrightarrow \alpha x$.

Definition 2.3.3. Let X be a quasi-2-normed space with power r, then a functional F over $X \times X$ is called a linear 2-functional if the following conditions are satisfied

(i) $F(a+c, b+d) = F(a,b)+F(a,d)+F(c,b)+F(c,d)$.

(ii) $F(\alpha a, \beta b) = \alpha\beta F(a,b)$, for $a,b,c,d \in X$ and α, β are scalars.

Definition 2.3.4. A linear 2-functional is called 2-continuous at the point (a,b) if given $\epsilon > o$ there exists $\delta > o$ such that $\| F(a,b) - F(a,b) \|_s < \epsilon$ when ever $\nu(a-c, b) < \delta$ and $\nu(c, b-d) < \delta$, or $\nu(a-c,d)<\delta$ and $\nu(a,b-d)<\delta$

F is 2-continuous if it is continuous at each points of its domain.

Definition 2.3.5. Let F be a 2-functional with domain $D(F)$. F is called bounded if there is a real constant $K \geq 0$ such that $|F(a,b)| \leq K\, \nu(a,b)$ for all $(a,b) \in D(F)$. If F is bounded define the norm of F, $\|F\|$, by

$$\|F\| = \text{glb}\{K : |F(a,b)| \leq K \quad (a,b)$$

$$\text{for all } (a,b) \in D(F)\}.$$

If F is not bounded define $\|F\| = +\infty$.

Slightly modefying the proofs of Theorem 2.5 and Theorem 2.7 [55], we get the following theorems :

<u>Theorem</u> 2.3.2. The space of all bounded linear 2-functionals defined on a 2-Banach space B is a Banach space.

<u>Theorem</u> 2.3.3. Let $(B, \nu_p(\cdot,\cdot))$ be a complete quasi-2-normed space and let M and $[b]$ be linear manifolds in B. Let F be a bounded linear 2-functional with domain $M \times [b]$. Then there exists a bounded linear 2-functional H with domain $B \times [b]$ such that :

(i) $H(a,\alpha b) = F(a,\alpha b)$ for all $(a,\alpha b) \in M \times [b]$

(ii) $\|H\|_p = \|F\|_p$.

CHAPTER - III

QUASI 2-NORMED LATTICE

In the present chapter, we study the concept of quasi-2-normed lattices. It may be remarked that the concept of quasi-normed lattice has been studied by A. H. Siddiqi and M. A. Shahabi [52].

<u>Definition</u> 3.1. [54]. A real linear space X is called a Riesz space (or K-linear or linear lattice or vector lattice or lattice ordered vector space) if in X it is indicated which elements are considered to be greater than zero, and the following axioms are satisfied.

(1) If $x > o$, then $x \neq o$.

(2) If $x > o$, and $y > o$, then $x+y > o$.

(3) For any two elements $x, y \in X$ their supremum $x \vee y$ exists.

(4) If $x > o$, then for number $\lambda > o$, $\lambda x > o$.

<u>Definition</u> 3.2. A linear lattice (Riesz space) X is called quasi-2-normed lattice (quasi-2-normed Riesz space) with power p if a real-valued function ν is defined over $X \times X$ with the following properties.

(i) $\nu(x,y) \geq o$, $\nu(x,y) = o$ if, and only if x and y are linearly dependent.

(ii) $\nu(x,y) = \nu(y,x)$.

(iii) $\nu(x,\alpha y) = |\alpha|^p \; \nu(x,y)$ where α is scalar and $o < p \leq 1$.

(iv) $\nu(x,y+z) \leq \nu(x,y) + \nu(x,z)$ for all x,y and $z \in X$.

(v) If $|x| \leq |y|$ and $|z| \leq |z'|$, then $\nu(x,z) \leq \nu(y,z')$.

If a quasi-2-normed lattice is complete with respect to every semi norm $\mu_b(a) = \nu(a,b)$ for $b \in X$, then X is called quasi-2-Banach lattice.

Definition 3.3. [54]. A Riesz space X is called Archimedean if the axiom of Archimedes is satisfied in it 'from the fact that for some $x \geq o$ ($x \in X$) the set $\{nx\}_{n=1,2,\ldots}$ of all its multiples is bounded the equality $x = o$ follows'.

Definition 3.4. [54]. A Dedekind complete Riesz space in which there is defined a monotonic norm is called a Dedekind complete normed lattice (or a KN-space). In other words, a KN-space is a normed lattice which is Dedekind complete as a lattice.

__Theorem__ 3.1. The axiom of Archimedean is satisfied in every quasi-2-normed lattice.

__Proof__. Let $x \geq o$ and $nx \leq y$ for all $n = 1,2,3,\ldots$. Since $z \leq z$ for every $z \in X$, we have

$$n^p \nu(x,z) = \nu(nx,z) \leq \nu(y,z).$$

Or $\nu(x,z) \leq \frac{1}{n^p}\nu(y,z)$ for all n. This implies that $\nu(x,z) = o$, which means that x is linearly dependent to all $z \in X$, that is $x = o$.

__Theorem__ 3.2. Let $(X_1, \nu_1), (X_2, \nu_2), \ldots, (X_n, \nu_n)$ be quasi-2-normed lattices with power $p_1, p_2, \ldots, p_n$ respectively, then $(X = \prod_{i=1}^{n} X_i, \nu i)$ is a quasi-2-normed lattice with the power $p_1 p_2 \ldots p_n$, where

$$\nu(x,y) = (\nu_1(x_1,y_1))^{a_1} + (\nu_2(x_2,y_2))^{a_2} + \ldots + (\nu_n(x_n,y_n))^{a_n},$$

and

$a_i = p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_n$. If $(X_1, \nu_1), (X_2, \nu_2), \ldots, (X_n, \nu_n)$ are quasi 2-Banach lattice then (X, ν) is quasi-2-Banach lattice.

<u>Proof.</u> X is a Riesz space with the following relation.

For $x = (x_1, x_2, \ldots, x_n) \in X$, $y = (y_1, y_2, \ldots, y_n) \in X$, $x \leq y$ if $x_i \leq y_i$ for $i = 1, 2, \ldots, n$.

(i) If $\nu(x,y) = o$, then $(\nu_i(x_i,y_i))^{s_i} = o$ for $i = 1, 2, \ldots, n$ and so $\nu_i(x_i,y_i) = o$. This implies that x_i and y_i are linearly dependent for $i = 1, 2, \ldots, n$ and consequently $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are linearly dependent.

Conversely, let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are linearly dependent. Then x_i and y_i are linearly dependent for $i = 1, 2, \ldots, n$, which implies

$$\nu_i(x_i,y_i) = o.$$

Or $\qquad (\nu_i(x_i,y_i))^{s_i} = o$ for $i = 1, 2, \ldots, n$ and

hence $\qquad \nu(x,y) = o.$

Therefore $\nu(x,y) = o$ if, and only if x and y are linearly dependent.

(ii) $\nu(x,y) = \nu(y,x)$ is obvious.

(iii) $\nu(x,\alpha y) = (\nu_1(x_1,\alpha y_1))^{s_1} + (\nu_2(x_2,\alpha y_2))^{s_2} + \ldots$

$$+ (\nu_n(x_n,\alpha y_n))^{s_n} .$$

$$= (|\alpha|^{p_1} \nu_1(x_1,y_1)) + (|\alpha|^{p_2} \nu_2(x_2,y_2)) + \ldots$$

$$+ (|\alpha|^{p_n} \nu_n(x_n,y_n)) .$$

$$= |\alpha|^{p_1 p_2 \ldots p_n} (\nu_1(x_1,y_1))^{s_1} + \ldots$$

$$+ (\nu_n(x_n,y_n))^{s_n} ,$$

$$= |\alpha|^{p_1 p_2 \ldots p_n} \nu(x,y).$$

(iv) $\nu(x,y+z) = (\nu_1(x_1,y_1+z_1))^{s_1} + (\nu_2(x_2,y_2+z_2))^{s_2} + \ldots$

$$+ (\nu_n(x_n,y_n+z_n))^{s_n}$$

$$\leq (\nu_1(x_1,y_1) + \nu_1(x_1,z_1))^{s_1} + (\nu_2(x_2,y_2) +$$

$$+ {}_2(x_2,z_2))^{s_2} + \ldots + (\nu_n(x_n,y_n) +$$

$$+ \nu_n(x_n,z_n))^{s_n} ,$$

$$\leq (\nu_1(x_1,y_1))^{s_1} + (\nu_1(x_1,z_1))^{s_1} + (\nu_2(x_2,y_2))^{s_2} + (\nu_2(x_2,z_2))^{s_2} + \dots + (\nu_n(x_n,y_n))^{s_n} + (\nu_n(x_n,z_n))^{s_n}$$

(by the inequality $|a+b|^p \leq |a|^p + |b|^p$)

$$= \nu(x,y) + \nu(x,z).$$

Hence $\nu(x,y+z) \leq \nu(x,y) + \nu(x,z)$.

(v) Let $|x| \leq |y|$ and $|z| \leq |z'|$, then $|x_i| \leq |y_i|$ and $|z_i| \leq |z_i'|$.

Now

$$\nu(x,Z) = (\nu_1(x_1,z_1))^{s_1} + (\nu_2(x_2,z_2))^{s_2} + \dots + (\nu_n(x_n,z_n))^{s_n},$$

$$\leq (\nu_1(y_1,z_1'))^{s_1} + (\nu_2(y_2,z_2'))^{s_2} + \dots + (\nu_n(y_n,z_n'))^{s_n}$$

$$= \nu(y,z').$$

Hence

$$\nu(x,z) \leq \nu(y,z').$$

If (X_i, ν_i), $i = 1,2,\ldots,n$ are quasi-2-Banach lattices, that is for every sequence $\{x_i^m\}$ in X_i satisfying

$$\lim_{\substack{m\to\infty \\ m'\to\infty}} \mu_{i_{b_i}}(x_i^m - x_i^{m'}) = o ,$$

we have

$$\lim_{m\to\infty} \mu_{i_{b_i}}(x_i^m - x_i) = o ,$$

for $x_i \in X_i$ and for all $b_i \in X_i$, then it is clear that for every sequence $\{x^m\}$ in X with $x^m = (x_1^m, x_2^m, \ldots, x_n^m)$ satisfying

$$\lim_{m\to\infty} \mu_b(x_m - x_{m'}) = o \text{ for all } b = (b_1, b_2, \ldots, b_n),$$

we have

$$\lim_{m\to\infty} \mu_b(x_m - x) = o \quad \text{for } x \in X \text{ with } x = (x_1, \ldots, x_n).$$

Thus (X, ν) is a complete with respect to each semi-norm $\mu_b(x) = \nu(x, b)$. Therefore (X, ν) is a 2-Banach lattice.

<u>Theorem</u> 3.3. Let X be a quasi-2-normed lattice and x_n, y_n, x, y and $z \in X$. If $\nu(x_n - x, z) \longrightarrow o$, $\nu(y_n - y, z) \longrightarrow o$ as $n \longrightarrow \infty$, then $\nu((x_n \vee y_n) - (x \vee y), z) \longrightarrow o$, $\nu((x_n \; y_n) - (x \; y), z) \longrightarrow o$ as $n \longrightarrow \infty$.

Proof. By the inequality $\nu((x \vee z) - (y \vee z), 1) \leq \nu(x-y, 1)$, we have

$$\nu((x_n \vee y_n)-(x \vee y), s) \leq \nu((x_n \vee y_n)-(x \vee y_n)+(x \vee y_n)-(x \vee y), s)$$

$$\leq \nu((x_n \vee y_n)-(x \vee y_n), s) +$$

$$+ \nu((x \vee y_n)-(x \vee y), s)$$

$$\leq \nu(x_n - x, s) + \nu(y_n - y, s).$$

i.e.

$$\nu((x_n \vee y_n)-(x \vee y), s) \leq \nu(x_n - x, s) + \nu(y_n - y, s).$$

Consequently

$$\nu((x_n \vee y_n) - (x \vee y), s) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Similarly, we can prove that

$$\nu((x_n \quad y_n) - (x \quad y), s) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Theorem 3.4. If X is a quasi-β-normed lattice and $\hat{X}$ is its Dedekind completion, then the quasi-β-norm defined in X can be extended to $\hat{X}$ so that $\hat{X}$ becomes a Dedekind complete quasi-β-normed lattice.

Proof. For arbitrary $a, b \in \hat{X}$ we set

$$\nu(a,b)^* = \inf_{\substack{c,d \in X \\ c \geq |a|, d \geq |b|}} \nu(c,d)$$

Clearly $\nu(a,b)^* = \nu(a,b)$ for $a, b \in X$ since the quasi-2-norm defined in X is monotone. We shall verify that $\nu(a,b)^*$ satisfies all axioms of quasi-2-norm in $\hat{X}$.

$$\nu(\lambda a,b)^* = \inf_{\substack{c,d \in X \\ c \geq |a|, d \geq |b|}} \nu(\lambda c,d) = \inf_{\substack{c,d \in X \\ c \geq |c|, d \geq |b|}} |\lambda|^p \nu(c,d)$$

$$= |\lambda|^p \inf_{\substack{c,d \in X \\ c \geq |a|, d \geq |b|}} \nu(c,d) = |\lambda|^p \nu(a,b)^*.$$

If $a, b \in \hat{X}$, then there exists $c, d \in X$ such that

$$c \leq |a|, \quad d \leq |b|$$

and $$\nu(a,b)^* \geq \nu(c,d) \geq 0.$$

Let $a = a_1 + a_2$, $b \in \hat{X}$. If $|a_1| \leq c_1$, $|a_2| \leq c_2$, $|b| \leq d$, then $|a| \leq c_1 + c_2$.

Therefore

$$\nu(a,b)^* \leq \inf_{\substack{c_1, c_2, d \in X \\ c_1 \geq |a|, c_2 \geq |b|}} \nu(c_1 + c_2, d) ,$$

$$\leq \inf_{\substack{c_1, c_2 \in X \\ c_1 \geq |a|, c_2 \geq |b|}} \left\{ \nu(c,d) + \nu(c_2,d) \right\}$$

$$= \inf_{\substack{c_1, d \in X \\ c_1 \geq |a|, d \geq |b|}} \nu(c_1,d) + \inf_{\substack{c_2, d \in X \\ c_2 \geq |a|, d \geq |b|}} \nu(c_2,d)$$

$$= \nu(a_1,d)^* + \nu(a_2,d)^*.$$

Thus, $\nu(a,b)^*$ is a quasi-2-norm. Its monotonicity is obvious, consequently, $\hat{X}$ is Dedekind complete 2-normed lattice.

CHAPTER - IV

NON-ARCHIMEDEAN QUASI-2-NORMED SPACE

In the present chapter, we discuss the results concerning non-archimedean quasi-2-normed spaces. It may be observed that non-archimedean quasi-normed spaces have been studied by Honda [41]. There are two sections in this chapter. Section 3.1, deals with the definition and properties of non-archimedean quasi-2-normed spaces while in Section 3.2, we discuss the behaviour of 2-operators on non-archimedean quasi-2-normed space.

4.1. DEFINITION AND PROPERTIES OF NON-ARCHIMEDEAN QUASI-2-NORMED SPACES

Definition 4.1.1. Let X be a vector space of dimension greater than 1 over field K with non-trivial non-archimedean valuation. A real valued function $\nu(x,y)$ over $X \times X$ is called a non-archimedean quasi-2-norm with power p, $o < p \leq 1$ if it satisfies the following conditions.

(i) $\nu(x,y) = o$ if, and only if x and y are linearly dependent,

(ii) $\nu(x, \beta y) = |\beta|^p \nu(x,y)$ for $\beta \in K$,

(iii) $\nu(x,y) = \nu(y,x)$ for all $x,y \in X$,

(iv) $\nu(x,y+z) \leq \max(\nu(x,y), \nu(x,z))$ for all $x,y,z \in X$.

$(X,\nu(.,.))$ is called non-archimedean quasi 2-normed space.

<u>Remark.</u> For $p = 1$, we get non-archimedean 2-normed space.

<u>Theorem</u> 4.1.1. In the above definition condition (iv) is equivalent to the following.

(iv)' $\nu(x+z,y+z) \leq \max(\nu(x,y), \nu(y,z), \nu(z,x))$.

That is, non-archimedean quasi-2-norm satisfies (iv)'.

<u>Proof.</u> Suppose ν is a non-archimedean quasi-2-norm, then by (iv) of the above definition, we get

$$\nu(x+z,y+z) \leq \max(\nu(x+z,y), \nu(x+z,z))$$

$$\leq \max(\nu(x,y), \nu(z,y), \nu(x,z), \nu(z,z)).$$

Hence

$$\nu(x+z,y+z) \leq \max(\nu(x,y), \nu(z,y), \nu(x,z)) \text{ because } \nu(z,z) = 0.$$

Now we suppose that (i), (ii), (iii) and (iv) hold. Then

$$\nu(x,y+z) = \nu(-\alpha z+(x+\alpha z),(y+z-x-\alpha z)+(x+\alpha z))$$

$$\leq \max \ (\nu(-\alpha z,y+z-x-\alpha z),\ \nu(y+z-x-\alpha z,x+\alpha z),$$

$$\nu(x+\alpha z,-\alpha z))$$

$$\leq \max\ (\nu(x+\alpha z,y+z),\ |\alpha|^{p}\{\nu(y-x,z),\ \nu(x,z)\}).$$

Since,

$$\nu(x+\alpha z,\alpha y+\alpha z) \leq \max\ (\nu(x,\alpha y),\ \nu(\alpha y,\alpha z),\ \nu(\alpha z,x))$$

(by (iv)').

We get

$$|\alpha|^{p}\nu(x+z,y+z) \leq \max\ (|\alpha|^{p}\nu(x,y),\ |\alpha|^{2p}\nu(y,z),$$

$$|\alpha|^{p}\nu(z,x)) \quad \text{(by (i))}$$

Or

$$\nu(x+\alpha z,y+z) \leq \max\ (\nu(x,y),\ |\alpha|^{p}\nu(y,z),\ \nu(x,z)),$$

for $\alpha \neq o$.

Thus

$$\nu(x,y+z) \leq \max\ (\nu(x,y),\ \nu(x,z),\ |\alpha|^{p}\nu(y,z),$$

$$\{\nu(y-x,z),\ \nu(x,z)\}).$$

By taking $\alpha \longrightarrow o$, we have

$$\nu(x,y+z) \leq \max\ (\nu(x,y), \nu(x,z)) .$$

This completes the proof of the theorem.

<u>Theorem</u> 4.1.2. Let X be a non-archimedean quasi-2-normed space and α be any real number, then

$$\nu(x,y) = \nu(x,y+\alpha x).$$

<u>Proof.</u> For any α and any points $x,y \in X$, we have

$$\nu(x,y) = \nu(x,\ y+\alpha x - \alpha x\ )$$

$$\leq \max\ (\nu(x,y+\alpha x), \nu(x,-\alpha x)), \quad \text{(by (iv))}$$

$$= \nu(x,y+\alpha x), \text{ as } \nu(x,-\alpha x) = o \quad \text{(by (i))}$$

and

$$\nu(x,y+\alpha x) \leq \max\ (\nu(x,y), \nu(x,\alpha x)$$

$$= \nu(x,y).$$

Hence

$$\nu(x,y) = \nu(x,y+\alpha x).$$

<u>Theorem</u> 4.1.3. Every non-archimedean quasi-2-normed space X is a locally K-convex topological vector space.

Proof. Let U be a given neighbourhood of $a-b$. By Theorem 2.1 [14], there is a neighbourhood in U of $a-b$ of the form $W_{\Sigma(a-b)}$, $\Sigma = \{(b_1, e_1), \ldots, (b_n, e_n)\}$.
$\Sigma_b = \{(a-b_1, e_1), \ldots, (a-b_n, e_n)\}$. For $i = 1, 2, \ldots, n$ and $a' \in W_{\Sigma_a}(a)$, where $\Sigma_a = \{(b_1+b, e_1), \ldots, (b_n+b, e_n)\}$,
$\nu(b_i-a+b, a'-a) = \sigma(a, a_i+b, a') < e_i$, where σ is the ultra 2-metric induced by non-archimedean quasi-2-norm. For every $b' \in W_{\Sigma_b}(b)$, $\nu(b_i-a+b, b'-b) = \nu(-b_i+a-b, b'-b) = \sigma(b, a_i-b, b') < e_i$, so we have

$$\begin{aligned}
\sigma(a-b, b_i, a'-b') &= \nu(b_i-a+b, a'-b'-a+b) \\
&\leq \max(\nu(b_i-a+b, a'-a), \\
&\qquad \nu(b_i-a+b', b'-b)) \\
&\leq \max(e_i, e_i) = e_i.
\end{aligned}$$

That is for every point $a' \in W_{\Sigma_a}(a)$ and every point $b' \in W_{\Sigma_b}(b)$, $a'-b' \in W_{\Sigma(a-b)} \subseteq U$. Therefore the mapping $(a, b) \longrightarrow a-b$ over $X \times X$ into X is continuous.

Now we show that the mapping $(\alpha, a) \longrightarrow \alpha a$ of $R \times X$ into X is continuous. Let for $\alpha \in R$ and $a \in X$, U be a neighbourhood of a. By Theorem 2.1 [14] there is in U a neighbourhood of a of the form $W_{\Sigma}(\alpha a)$ with

$$\Sigma = \{(b_1, e_1), \ldots, (b_n, e_n)\}.$$

Let

$$m_i = \begin{cases} \dfrac{e_i}{(a, b_i)} & \text{for} \quad (a, b_i) \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

$$m = \inf m_i, \quad e_i' = \frac{e_i}{\{|\alpha|^p + m\}} \quad \text{and}$$

$$\Sigma' = \{(b_1 + a - \alpha a, e_1'), \ldots, (b_n + a - \alpha a, e_n')\}.$$

Then for every real number with $|\alpha - \alpha'|^p < m$ and every point $a' \in W_{\Sigma'}(a)$.

$$\begin{aligned} \sigma(\alpha a, b_i, \alpha' a') &= \nu(b_i - \alpha a, \alpha' a' - \alpha a) \\ &= \nu(b_i - \alpha a, (\alpha' a' - \alpha' a) + (\alpha' a - \alpha a)) \\ &\leq \max(\nu(b_i - \alpha a, (\alpha' a' - \alpha' a)), \\ &\qquad \nu(b_i - \alpha a, \alpha' a - \alpha a)) \\ &\leq \max(|\alpha'|^p \sigma(a, b_i + a - \alpha a, a'), |\alpha' - \alpha|^p \nu(b_i, a)) \end{aligned}$$

$$\leq \max(\{|\alpha|^p + m\} \, e_i, \, m_i \, \nu(a, b_i))$$

$$\leq \max(e_i, e_i)$$

$$= e_i, \; i = 1, 2, \ldots, n.$$

This implies that $\alpha' a' \in V_{\Sigma}(\alpha a) \subseteq U$. Thus the mapping $(\alpha, a) \longrightarrow \alpha a$ is continuous.

In the last we show that a fundamental system of neighbourhoods $V_{\Sigma}(o)$ of o consists of only K-convex sets. For $\Sigma = \{(b_1, e_1), \ldots, (b_n, e_n)\}$ suppose the points a and b belong to the set $V_{\Sigma}(o)$. For $i = 1, 2, \ldots, n$ and $\alpha \in K$ $|\alpha| \leq 1$

$$\sigma(o, b_i, \alpha a + (1-\alpha)b) = \nu(b_i, \alpha a + (1-\alpha)b)$$

$$\leq \max(|\alpha|^p \nu(o, b_i, a), \; |1-\alpha|^p \sigma(o, b_i, b))$$

$$< e_i.$$

This implies that $\alpha a + (1-\alpha)b \in V_{\Sigma}(o)$. Hence $V_{\Sigma}(o)$ is K convex. This completes the proof of the theorem.

<u>Theorem</u> 4.1.4. Every non-archimedean quasi-2-normed space is a ultra 2-metric space.

<u>Proof</u>.1. On the basis of the assumption that the space has a dim $\neq$ 1, there always exists a point c to two arbitrary points a,b that are different from one another so that the vectors b-a and c-a are linearly independent, that is

$$\sigma(a,b,c) = \nu(b-a,\ c-a) \neq 0\ .$$

Further

$$\sigma(a,b,c) = \nu(b-a,\ c-a) = 0$$

is true if atleast two of the three points a,b,c are equal.

2. For arbitrary points a,b and c of L

$$\nu(b-a,\ c-a) = \nu(c-a,\ b-a) \qquad ..\ (1)$$

Further

$$\nu(c-a,b-a) = \nu((c-b)-(a-b),-(a-b))$$

$$= \nu(c-b,\ a-b) \qquad ..\ (2)$$

From the equation (1) and (2) we get

$$\sigma(a,b,c) = \sigma(a,c,b) = \sigma(b,c,a).$$

3. It is true finally

$$\sigma(a,b,c) = \nu(b-a,c-a)$$

$$= \nu((b-d)-(a-d),(c-d)-(a-d))$$

$$\leq \max(\nu(b-d,c-d), \nu(c-d,a-d), \nu(a-d,b-d))$$

$$= \max(\sigma(a,b,d), \sigma(a,d,c), \sigma(d,b,c)).$$

This completes the proof of the theorem.

4.2. BEHAVIOUR OF 2-OPERATORS ON NON-ARCHIMEDEAN QUASI-2-NORMED SPACES

Definition 4.2.1. Let $(B,\nu(.,.))$ be a non-archimedean quasi-2-normed space with power r and $(C,\mu(.))$ be a non-archimedean quasi-normed space with power s. B and C are defined over the same field K complete with non-trivial valuation. Then an operator T over $B \times B$ into C is called a linear 2-operator if the following conditions are satisfies.

(i) $T(x+y,w+z) = T(x,w) + T(x,z) + T(y,w) + T(y,z)$,

(ii) $T(\alpha x+\beta y) = \alpha\beta T(x,y)$ for all $\alpha,\beta \in K$ and x,y,z and $w \in B$.

If $C = R$ (space of real numbers) then T is called 2-functional.

__Definition__ 4.2.2. A linear 2-operator T is called 2-continuous at the point $(x,y) \in B \times B$ if for given $\epsilon > o$ there exists $\delta > o$ such that $\mu_B(T(x,y) - T(w,z)) < \epsilon$ whenever $\nu(x-w,y) < \delta$ and $\nu(w,y-z) < \delta$ or $\nu(x-w,z) < \delta$ and $\nu(x,y-z) < \delta$. T is 2-continuous if it continuous at each points of its domain.

__Theorem__ 4.2.1. If a linear 2-operator T is 2-continuous at (o,o) then it is 2-continuous.

__Proof__. Since T is linear and 2-continuous, we have $T(o,o) = o$ and for given $\epsilon > o$, there is a $\delta > o$ such that $\mu_B(T(x,y)) < \epsilon$ whenever $(x,y) < \delta$. For an arbitrary point $(w,z) \in B \times B$, we have

$$\begin{aligned}\mu_B(T(w,z)-T(x,y)) &= \mu_B(T(w,z)-T(x,z)+T(x,z)-T(x,y)) \\ &= \mu_B(T(w-x,z) + T(x,z-y)) \\ &\leq \max\{\mu_B(T(w-x,z)),\mu_B(T(x,z-y))\} \\ &< \epsilon, \text{ whenever } \nu(w-x,z)<\delta \text{ and } \nu(x,z-y)<\delta.\end{aligned}$$

Hence T is 2-continuous at any arbitrary point (w,s). This proves the theorem.

<u>Theorem</u> 4.2.2. Non-archimedean quasi-2-norm on a linear space B is continuous 2-functional.

<u>Proof</u>. We have

$$
\begin{aligned}
\nu_s(x,y) = \nu_s(x-s+s,y) &\le \max\{\nu_s(x-s,y),\ \nu_s(s,y)\} \\
&\le \max\{\nu_s(x-s,y), \nu_s(s,y-w), \\
&\qquad\qquad \nu_s(s,w)\} \\
&\le \nu_s(x-s,y) + \nu_s(s,y-w) + \nu_s(s,w),
\end{aligned}
$$

so

$$\nu_s(x,y) - \nu_s(s,w) \qquad \le \nu_s(x-s,y) + \nu_s(s,y-w).$$

Similarly we can show that

$$\nu_s(s,w) - \nu_s(x,y) \le \nu_s(x-s,y) + \nu_s(s,y-w).$$

Therefore

$$|\nu_s(s,w) - \nu_s(x,y)| \le \nu_s(x-s,y) + \nu_s(s,y-w).$$

This implies that ν_s is continuous.

Definition 4.2.3. T is said to be bounded if there is a real constant M such that

$$\mu_s(T(x,y)) \leq M(\nu_r(x,y))^{s/r} \text{ for all } x,y \in B.$$

If T is bounded, we define the norm $\bar{\mu}$ of T by the relation

$$\bar{\mu}_s(T) = \inf\{ M: \mu_s(T(x,y)) \leq M(\nu_r(x,y))^{s/r} \} .$$

If T is unbounded, then $\bar{\mu}_s(T) = +\infty$.

Now we prove the following theorem.

Theorem 4.2.3. A linear 2-operator T is bounded if, and only if it is continuous.

In the proof of the Theorem 4.2.3, we require the following lemma.

Lemma 4.2.1. If T is continuous at (o,o), then for every sequence $\{x_n\}$ converging to o in B, $\{T(x_n,z)\}$ converges to o in C for all $z \in B$.

Proof of Lemma 4.2.1. Since T is continuous at (o,o), for given $\epsilon > o$ there is a $\delta > o$ such that $\mu_s(T(x,y)) < \epsilon$ whenever $\nu(x,y) < \delta$. Since $\{x_n\}$ converges to o, for $\delta > o$, we can find a number N such that $\nu(x_n,z) < \delta$ for

$n \geq N$ and all $s \in B$. Thus we have

$$\mu_s(T(x_n,s)) < \varepsilon \quad \text{for} \quad n \geq N \text{ and } s \in B.$$

Hence $\{T(x_n,s)\}$ converges to 0 in C.

<u>Proof of Theorem 4.2.3</u>. Let T be continuous but not bounded, that is to say that there does not exist a constant M such that

$$\mu_s(T(x,y)) \leq M\,(\nu_r(x,y))^{s/r}, \text{ for all } (x,y) \in B \times B.$$

Now we consider two cases :

(a) Suppose that the valuation of K is discrete. Then $K_r = \{\rho^i - \infty < i < \infty ; \rho > 1\}$. Let there exists a sequence x_n in B and a sequences of numbers M_n with $M_n \longrightarrow +\infty$ such that

$$\mu_s(T(x_n,y)) > M_n(\nu_r(x_n,y))^{s/r}, \; y \in B.$$

Let a_n be elements of K such that

$$\frac{1}{\rho}\,\frac{1}{M_n(\nu_r(x_n,y))^{s/r}} \leq |a_n|^s \leq \frac{1}{M_n(\nu_r(x_n,y))^{s/r}}$$

and let $z \in K_r$ be the smallest of numbers which are greater than or equal to

$$\frac{1}{\rho}\,\frac{1}{M_n(\nu_r(x_n,y))^{s/r}}\text{, for some } n\text{. Then } \alpha \geq \frac{1}{M_n(\nu_r(x_n,y))^{s/r}}$$

and $\dfrac{\alpha}{\rho} < \dfrac{1}{\rho M_n(\nu_r(x_n,y))^{s/r}}$, that is, $\dfrac{1}{M_n(\nu_r(x_n,y))^{s/r}} > \alpha.$

Hence there will be some $\alpha_n \in K$ such that $|\alpha_n|^s = \alpha$.

Since $(\nu_r(x_n,y))^{s/r} \neq 0$, let $z_n = \alpha_n x_n$, then

$$(\nu_r(z_n,y))^{s/r} = (\nu_r(\alpha_n x_n,y))^{s/r} = |\alpha_n|^s (\nu_r(x_n,y))^{s/r}$$

$$\leq \frac{1}{M_n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

This implies that $\{z_n\}$ converges to 0 in E. On the other hand,

$$\mu_s(T(z_n,z)) = \mu_s(T(\alpha_n x_n,z))$$

$$= |\alpha_n|^s \mu_s(T(x_n,z))$$

$$\geq \frac{1}{\rho}\,\frac{1}{M_n}\,\frac{\mu_s(T(x_n,z))}{(\nu_r(x_n,y))^{s/r}} \geq \frac{1}{\rho},$$

which means that $T(z_n,z)$ can not converge to 0 in G. Hence by virtue of Lemma 4.2.1, T can not be continuous at $(0,0)$, which is a contradiction. This completes proof in this case.

(b) Let the valuation of K be non-discrete. Then N_K is everywhere dense in the field of real numbers. Hence for every $\varepsilon > 0$, there exists elements of K such that

$$\frac{1}{M_n(\nu_r(x_n,y))^{s/r}} - \varepsilon \leq |\alpha_n|^s \leq \frac{1}{M_n(\nu_r(x_n,y))^{s/r}} .$$

Now proceeding as in case (a), we get the result.

The necessary part follows immediately as it can be seen easily that bounded linear 2-operator is 2-continuous at (o,o) and hence 2-continuous every where in the domain by Theorem 4.2.1. This completes the proof of the theorem.

Let B and C are as in Definition 4.2.1, and $[B,C]$ denotes the space of all bounded linear 2-operators from $B \times B$ into C, with dim $C > 1$.

Theorem 4.2.4. $([B,C],\|\ \|)$ is a complete non-archimedean normed linear space if the space C is complete.

Proof. It is easy to verify that $S+T$ and αS belong to $[B,C]$, for all $S, T \in [B,C]$ and $\alpha \in K$, with respect to these operations of addition and scalar multiplication.

$[B,C]$ is a linear space over the field K. We define

$$\bar{\mu}(T) = \sup_{(\nu_r(x,y))^{s/r} \neq 0} \frac{\mu_s(T(x,y))}{(\nu_r(x,y))^{s/r}} .$$

The following conditions are satisfied.

(1) If $\bar{\mu}(T) = 0$, then $\mu_s(T(x,y)) = 0$ for all $x,y \in B$ or $T(x,y) = 0$ for all $x,y \in B$, hence $T = 0$.

If $T = 0$, then $T(x,y) = 0$ for all $x,y \in B$ and hence

$$\bar{\mu}_s(T) = \sup_{(\nu_r(x,y))^{s/r} \neq 0} \frac{\mu_s(T(x,y))}{(\nu_r(x,y))^{s/r}} = 0.$$

(2) $$\bar{\mu}_s(\alpha T) = \sup_{(\nu_r(x,y))^{s/r} \neq 0} \frac{\mu_s(\alpha T(x,y))}{(\nu_r(x,y))^{s/r}}$$

$$= |\alpha|^s \sup_{(\nu_r(x,y))^{s/r} \neq 0} \frac{\mu_s(T(x,y))}{(\nu_r(x,y))^{s/r}}$$

$$= |\alpha|^p \mu_s(T).$$

(3) $$\bar{\mu}_s(T+S) = \sup_{(\nu_r(x,y))^{s/r} \neq 0} \frac{\mu_s((S+T)(x,y))}{(\nu_r(x,y))^{s/r}}$$

$$= \sup_{(\nu_r(x,y))^{s/r} \neq 0} \frac{\mu_s(S(x,y)+T(x,y))}{(\nu_r(x,y))^{s/r}}$$

$$\leq \sup_{(\nu_r(x,y))^{s/r} \neq 0} \max \left\{ \frac{\mu_s(S(x,y))}{(\nu_r(x,y))^{s/r}}, \frac{\mu_s(T(x,y))}{(\nu_r(x,y))^{s/r}} \right\}$$

$$= \max \left\{ \sup_{(\nu_r(x,y))^{s/r} \neq 0} \frac{\mu_s(S(x,y))}{(\nu_r(x,y))^{s/r}}, \sup_{(\nu_r(x,y))^{s/r} \neq 0} \frac{\mu_s(T(x,y))}{(\nu_r(x,y))^{s/r}} \right\}$$

$$= \max \left\{ \bar{\mu}_s(S), \bar{\mu}_s(T) \right\}.$$

Hence $\bar{\mu}_s$ is a non-archimedean quasi-norm on $[B,C]$. Now let $\{T_n\}$ be a Cauchy sequence in $[B,C]$ that is ,

$$\lim_{mn \to \infty} \bar{\mu}_s(T_n - T_m) = 0.$$

Since

$$\bar{\mu}_s(T_n(x,y) - T_m(x,y)) \leq \bar{\mu}_s(T_n - T_m)(\nu_r(x,y))^{s/r},$$

therefore $\{T_n(x,y)\}$ is a Cauchy sequence in the complete non-archimedean quasi-normed space C and hence it is convergent.

Let us assume that $T(x,y) = \lim_n T_n(x,y)$. It is easy to show that T is bounded linear 2-operator. We shall show that $\lim_n T_n = T$.

For $(x,y) \in B \times B$, we define

$$S_{n,p}(x,y) = T_{n+p}(x,y) - T_n(x,y).$$

For given $\epsilon > 0$, there exists $N(\epsilon)$ such that for $(x,y) \in B \times B$, $n > N(\epsilon)$ and all natural numbers p,

$$\begin{aligned}\mu_s(S_{n,p}(x,y)) &= \mu_s(T_{n+p}(x,y)-T_n(x,y))\\ &\leq \bar{\mu}_s(T_{n+p}-T_n)(\nu_r(x,y))^{s/r}\\ &< \epsilon\,(\nu_r(x,y))^{s/r}.\end{aligned}$$

With fixed n, we have for all $(x,y) \in B \times B$,

$$\lim_{p\to\infty} S_{n,p}(x,y) = T(x,y) - T_n(x,y).$$

Since $\quad \lim_{p\to\infty} S_{n,p}(x,y) = \mu_s \lim_{p\to\infty} (S_{n,p}(x,y))$,

we have for all $(x,y) \in B \times B$ and for $n > N(\epsilon)$,

$$\mu_n(T(x,y) - T_n(x,y)) < \varepsilon \, (\nu_r(x,y))^{s/r}.$$

Hence $\lim\limits_{n \to \infty} \bar{\mu}_s(T - T_n) = 0$.

This completes the proof.

CHAPTER - V

ULTRA m-METRIC SPACES

The m-dimensional ($m \geq 2$) analogue of metric space was introduced by Menger [42], as generalized metric space. It was further studied in detail by Gähler [18,19]. In the present chapter, we study the concept of ultra m-metric spaces. The case $m = 2$ has been studied by S. Gähler, A.H. Siddiqi and S. C. Gupta [23]. There are three sections, in this chapter. Section 5.1, deals with the definitions and properties of ultra m-metric spaces. Section 5.2 is devoted to the properties of topology induced by ultra-m-metric while section 5.3 contains the result concerning ultra m-metrizability.

5.1. ULTRA m-METRIC SPACE

Definition 5.1.1 [18]. Let X be a non-empty set, m a natural number and $M = \{0,1,\ldots, m\}$. We denote by X^M, the M^{th} product of X, the set of all system $\mathcal{A} = (a_i)_{i \in M} = (a_0, a_1, \ldots, a_m)$ of elements a_i of X, the components of $\mathcal{A}$. Let p be a permutation on M such that for $\mathcal{A} = (a_i)_{i \in M} \in X^M$, $p(\mathcal{A}) = (a_{p(i)})_{i \in M} \in X^M$. By $\mathcal{A}_{a_i \to a}$, we denote an element of X^M, in which i^{th} component of $\mathcal{A} \in X^M$ is replace by $a \in R$. By

$\mathcal{a}_{a_i \to a,\, a_{i'} \to a'}$ $(i \neq i')$, we denote an element of X^M, in which i^{th} and i'^{th} components of $\mathcal{a}$ are replaced by a and a' respectively. For every point $\mathcal{a}=(a_i)_{i\in M}$ = $(a_0, a_1, \ldots, a_m)$ of X^M there is a real number $\sigma(\mathcal{a})$ = $\sigma(a_0, a_1, \ldots, a_m)$ with the following conditions :

$M_{1a.}$ $\sigma(\mathcal{a}) = 0$, if atleast two components of $\mathcal{a}$ are real.

$M_{1b.}$ For two different points of X there exists a point $\mathcal{a} \in X^M$, having these as the components such that $\sigma(\mathcal{a}) \neq 0$.

$M_{2.}$ For each point $\mathcal{a} \in X^M$ and each permutation p on M

$$\sigma(\mathcal{a}) = \sigma(p(\mathcal{a})).$$

M_3. $\sigma(\mathcal{a}) \leq \sum_{i\in M} \sigma(\mathcal{a}_{a_i \to a})$.

The function σ satisfying the conditions M_{1a}, M_{1b}, M_2 and M_3 is called a m-metric on X and X together with M-metric σ is called m-metric space.

<u>Definition</u> 5.1.2 [18]. A m-metric space (X, σ) is called

ultra n-metric space if the following condition is satisfied.

$$M_3'. \quad \sigma(\mathfrak{a}) \leq \max_i \; (\sigma(\mathfrak{a}_{a_i \to a}))\,.$$

A function σ on X^N with properties M_{1a}', M_{1b}', M_2 and M_3', is an ultra n-metric on X if, and only if it is non-negative.

<u>Theorem</u> 5.1.1. A n-metric σ on X is an ultra n-metric if, and only if for any point $\mathfrak{a} = (a_i)_{i \in N} \in X^N$ and any $a \in X$, the two largest of the numbers $\sigma(\mathfrak{a})$, $\sigma(\mathfrak{a}_{a_i \to a})$, $i \in N$ are equal.

<u>Proof</u>. Let σ be a n-metric on X. If for any point $\mathfrak{a} = (a_i)_{i \in N} \in X^N$ and any $a \in X$, the two largest of the above mentioned numbers are equal, then clearly σ satisfies the condition M_3', which implies that σ is an ultra n-metric on X.

We now suppose that σ is an ultra n-metric on X. Let for $\mathfrak{a} = (a_i)_{i \in N} \in X^N$ and $a \in X$, the condition

$$\sigma(\mathfrak{a}) \leq \sigma(\mathfrak{a}_{a_0 \to a}) \leq \dots \leq \sigma(\mathfrak{a}_{a_{n-1} \to a}) \text{ be satisfied.}$$

Then the relation

$$\sigma(\mathcal{A}_{a_m \to a}) \leq \max\{\sigma(\mathcal{A}), \sigma(\mathcal{A}_{a_0 \to a}), \ldots, \sigma(\mathcal{A}_{a_{m-1} \to a})\}$$

$$= \sigma(\mathcal{A}_{a_{m-1} \to a}) \text{ holds.}$$

If $\sigma(\mathcal{A}_{a_{m-2} \to a}) < \sigma(\mathcal{A}_{a_{m-1} \to a})$, then we get

$$\sigma(\mathcal{A}_{a_{m-1} \to a}) \leq \max\{\sigma(\mathcal{A}), \sigma(\mathcal{A}_{a_0 \to a}), \ldots, \sigma(\mathcal{A}_{a_{m-2} \to a}), \sigma(\mathcal{A}_{a_m \to a})\}$$

$$\leq \max\{\sigma(\mathcal{A}_{a_{m-2} \to a}), \sigma(\mathcal{A}_{a_m \to a})\}.$$

$$= \sigma(\mathcal{A}_{a_m \to a})$$

Thus the two largest of the numbers are equal.

5.2. TOPOLOGY INDUCED BY ULTRA m-METRIC

Let k and l be two different elements in M and $M_{kl} = M - \{k, l\}$. M_{kl} is empty if $m = 1$. By $X^{M_{kl}}$ for $m = 1$ we understand a set consisting of one point. For any $a, a' \in X$ and every $\mathcal{A} \in X^{M_{kl}}$, we denote by $[a, a', \mathcal{A}]$, the point $(a_i)_{i \in M} \in X^M$ in which $a_k = a$, $a_l = a'$ and for $m > 1$, each component a_i, $i \in M_{kl}$, is equal to the corresponding component of $\mathcal{A}$. We write $[a, a', \mathcal{A}]$

for $(\sigma[a, a', \mathfrak{A}])$. Denote by G system of all non empty subsets of $X^{\aleph_{\alpha 1}} \times E_{\varrho}$. We write $\Sigma < \Sigma'$, $\Sigma <' \Sigma$ and $\Sigma <'' \Sigma'$ if between Σ and Σ' there exists a one valued function, such that for eelements of Σ and Σ' of the form $(\mathfrak{A}, e)$ and $(\mathfrak{A}, e')$, $e < e'$, $e \leq e'$ and $e \nleq e'$ respectively. For each $a \in X$ and each $\Sigma \in G$, we define

$$W_{\Sigma}(a) = \{a' \in X, \ \sigma[a, a', \mathfrak{A}] < e \ \text{ for all } (\mathfrak{A}, e) \in \Sigma\} \text{ and ,}$$

$$W'_{\Sigma}(a) = \{a' \in X \ , \ \sigma[a, a', \mathfrak{A}] \leq e \ \text{ for all } (\mathfrak{A}, e) \in \Sigma\}.$$

If $\Sigma \leq' \Sigma'$ the $W_{\Sigma}(a) \subseteq W_{\Sigma'}(a)$, $W'_{\Sigma}(a) \subseteq W'_{\Sigma'}(a)$ and $W_{\Sigma}(a) \subseteq W'_{\Sigma'}(a)$ holds.

If $\Sigma <$ ' then $W'_{\Sigma}(a) \subseteq W_{\Sigma'}(a)$. And for each system of finite number of $\Sigma_i \in G$, we have $W_{\cup \Sigma_i}(a) = \bigcap_i W_{\Sigma_i}(a)$ and $W'_{\cup \Sigma_i}(a) = \bigcap_i W'_{\Sigma_i}(a)$. In view of Theorem 13 and Theorem 15 of Gähler and discussion on page 186 [18]. The family of all sets $W_{\Sigma}(a)$, $\Sigma \in G$ for a given $a \in X$ is an open base in a. The topology obtained in this way, does not depend on the special choice of K and 1. We call this topology a natural topology or topology induced by σ .

__Theorem__ 5.2.1. Let a and b be any points of X. For a given $E=\{(\mathcal{a}_1,e_1), (\mathcal{a}_2, e_2),\ldots, (\mathcal{a}_n, e_n)\}$ and let

$E' = E \cup \{ (\mathcal{a}_{i_{b_j}\to b}, e_i)\ i = 1,2,\ldots, n\ ,\ j \in E_{Kl}\}$. Then

$b \in U_{E'}(a)$ if, and only if $U_{E'}(a) = U_{E'}(b)$ holds.

__Proof.__ If $U_{E'}(a) = U_{E'}(b)$, then clearly $b \in U_{E'}(a)$.

Now let $b \in U_{E'}(a)$. For any $c \in U_{E'}(a)$, we have

$\sigma[a,c,\mathcal{a}] < e_i$, $\sigma[a,c, \mathcal{a}_{i_{a_j}\to b}] < e_i$ or

$\sigma[b, c, \mathcal{a}_{i_{a_j}\to a}] = \sigma[a, c, \mathcal{a}_{i_{a_j}\to b}] < e_i$. $b \in U_{E'}(a)$

implies that $\sigma[a,b,\mathcal{a}_i] < e_i$ and $\sigma[a,b,\mathcal{a}_{i_{a_j}\to b}] < e_i$.

Thus

$$\sigma[b,c,\mathcal{a}_i] \leq \max\{\sigma[a,c,\mathcal{a}_i], \sigma[b,a,\mathcal{a}_i] ,$$

$$\sigma[b,c,\mathcal{a}_{i_{a_j}\to a}] < e_i\} \text{ and } \sigma[b,c,\mathcal{a}_{i_{a_j}\to b}] < e_i$$

holds. These imply that $c \in U_{E'}(b)$ and therefore

$W_{\Sigma'}(a) \subseteq W_{\Sigma'}(b)$. Since $b \in W_{\Sigma'}(a)$, $a \in W_{\Sigma'}(b)$ and we can see as above that $W_{\Sigma'}(b) \subseteq W_{\Sigma'}(a)$. Thus we have $W_{\Sigma'}(a) = W_{\Sigma'}(b)$. This completes the proof of the theorem.

Theorem 5.2.2. The sets $_{\Sigma}(a)$, $W_{\Sigma'}(a)$ and $W_{\Sigma'}(a) \setminus W_{\Sigma}(a)$ are both open and closed.

Proof. Every set $W_{\Sigma}(a)$, $\Sigma = \{(\mathcal{a}_1, e_1), \ldots, (\mathcal{a}_n, e_n)\}$ is open by Theorem 20 [16]. Let b be any point of $\overline{W_{\Sigma}(a)}$ and $\Sigma' = \Sigma \cup \{(\mathcal{a}_{i_{a_j} \to a}, e_i)$, $i = 1,2,\ldots, n$, $j \in N_{K1}\}$.

Clearly $W_{\Sigma'}(b) \cap W_{\Sigma}(a)$ is non empty and there exists $c \in W_{\Sigma'}(b) \cap W_{\Sigma}(a)$ which implies that

$$\sigma[a,c, \mathcal{a}_i] < e_i \qquad i=1,2,\ldots, n$$

$$\sigma[c,b, \mathcal{a}_i] < e_i \qquad i=1,2,\ldots, n$$

$$\sigma[c,b, \mathcal{a}_{i_{a_j} \to a}] < e_i \qquad i=1,2,\ldots, n, \ j \in N_{K1}$$

and since

$$\sigma[a,b, \mathcal{a}_{i_{a_j} \to a}] = \sigma[c,b, \mathcal{a}_{i_{a_j} \to a}] < e_i,$$

$$i=1,2,\ldots, n, \ j \in N_{K1}$$

hold.

Therefore, we get

$$\sigma[a,b,\ _i] \leq \max\{\sigma[c,b,\alpha_i],\ \sigma[a,c,\alpha_i],$$

$$\sigma[a,b,\alpha_{i_{a-c}}]\} < e_i,$$

$$i = 1,2,\dots,n,\ j \in N_{K1}$$

and thus $b \in N_\Sigma(a)$. This shows that $U_\Sigma(a)$ is also closed.

2. Every set $U'_\Sigma(a)$, $\Sigma = \{(\alpha_1, e_1),(\alpha_2, e_2),\dots,$ (α_n, e_n) is closed by Theorem 2 1 [18] : Let b be a point of $U'_\Sigma(a)$ and $\Sigma' = \Sigma \cup \{(\alpha_{i_{a_j-a}}, e_i)$, $i=1,\dots,n,$ $j \in N_{K1}\}$. For every $c \in U_{\Sigma'}(b)$,

$$\sigma[b,c,\alpha_i] < e_i$$

$$\sigma[b,c,\alpha_{i_{a_j-a}}] < e_i,\ i = 1,2,\dots,n,\ j \in N_{K1}$$

and

$$\sigma[a,c,\alpha_{i_{a_j-b}}] = \sigma[b,c,\alpha_{i_{a_j-a}}] < e_i,$$

$$i = 1,\dots,n, j \in N_{K1}$$

held. Therefore, we get

$$\sigma[a,c,\ a_i] \leq \max\{\sigma[b,c,\alpha_i],\ \sigma[a,b,\alpha_i],\ \sigma[a,c,\alpha_{i_{a_j-b}}]\}$$

$$\leq e_i$$

(since $b \in W'_{\Sigma}(a)$, $\sigma[a, b, \alpha_i] < e_i$, $i = 1, 2, \dots, n$). Thus $c \in W'_{\Sigma}(a)$. Therefore, $W_{\Sigma'}(b) \subseteq W'_{\Sigma}(a)$. This proves that $W'_{\Sigma}(a)$ is open.

3. By first two it is clear that $W'_{\Sigma}(a) \setminus W_{\Sigma}(a)$ is closed and open.

5.3. ULTRA m- METRIZABILITY

Theorem 5.3.1. Let (X, T) be a topological space with more than two points. If (X, T) is ultra-metrizable, then (X, T) is ultra m-metrizable. There exists an ultra m-metric σ on X compatible with the given topology T such that (X, σ) has the properties K and S.

Proof. Let σ^* be an ultra metric on X compatible with T. By Theorem 58 [19], there are points $a_0, \dots, a_m$, so that $\sigma^*(a_i, a_j) \neq 0$ for each pair of different indices i, $j \in N = \{0, 1, \dots, m\}$ and for each point $\alpha = (a_i)_{i \in N} \in X^N$,

$$\sigma(\alpha) = \inf_{\substack{i, j \in N \\ i \neq j}} \sigma^*(a_i, a_j)$$

defines a m-metric σ on X compatible with T, such that (X, T) has the properties K and S.

We prove that σ is an ultra m-metric. Let $a_0,\dots,a_{m+1}$ be any $m+2$ points of X with

$$\overset{*}{\sigma}(a_0,a_1) = \inf_{x,y\in\{a_0,a_1,\dots,a_m\}} \overset{*}{\sigma}(x,y).$$

For such points we get

$$\sigma(a_0,a_1,\dots,a_m) = \sigma(a_0,a_1,\dots,a_{m-1},a_{m+1})$$

$$= \overset{*}{\sigma}(a_0,a_1)$$

$$\leq \inf_{i=0,1,\dots,m-1}\{\sigma(\mathfrak{a}_{a_i}-a_{m+1})\}$$

If $\sigma^*(a_0,a_1) < \overset{*}{\sigma}(a_0,x)$ or $\sigma^*(a_0,a_1) < \overset{*}{\sigma}(a_1,x)$ for $x\in\{a_2,a_3,\dots,a_{m+1}\}$, then by 2.A. [48] the relation

$$\sigma^*(a_0,x) = \sigma^*(a_1,x) \text{ holds.}$$

Thus

$$\sigma^*(a_0,a_k) = \sigma^*(a_1,a_k),\ k\in\{2,3,\dots,m+1\}$$

which implies that the largest two of the numbers $\sigma(\mathfrak{a})$ and $\sigma(\mathfrak{a}_{a_i}-a_{m+1})$ are equal. By Theorem 5.1.1, σ is an ultra m-metric.

CHAPTER - VI

m-NORMED SPACES OVER VALUED FIELD

In the present chapter, we describe a structure known as m-normed space over a valued field which turns out to be a locally K-convex topological vector space.

6.1. DEFINITIONS AND EXAMPLES

Definition 6.1.1 [18]. Let X be a non-empty linear space over a non-trivial valued field K, m be a natural number and $N_m = \{1, \ldots, m\}$. $p(\bar{a})$, $\bar{a}_{a_i \to a}$, $\bar{a}_{a_i \to a, a_j \to a'}(i \neq j)$ are similarly defined as for m-metric space, $\bar{a} = (a_1, \ldots, a_m) \in X^{N_m}$. Let ν be a real valued function on X^{N_m} with the following conditions.

1. $\nu(\bar{a}) \neq 0$ if, and only if the components $a_1, a_2, \ldots, a_m$ are linearly dependent.

2. $\nu(\bar{a}) = \nu(p(\bar{a}))$.

3. $\nu(\bar{a}_{a_i \to \alpha a_i}) = |\alpha| \, \nu(\bar{a})$ for all $\alpha \in K$ and $i = 1, \ldots, m$.

4. For each $\bar{a} = (a_i)_{i \in N_m} \in X^{N_m}$ and points a_i', a_i'' of X with $a_i = a_i' + a_i''$.

$$\nu(\mathcal{a}) \leq \nu(\mathcal{a}_{a_i - a_i'}) + \nu(\mathcal{a}_{a_i - a_i''}).$$

ν is called m-norm on X and (X, ν) m-normed space.

4'. For each $\mathcal{a} = (a_i)_{i \in N_o} \in X^{N_o}$ and points a_i', a_i'' of X with $a_i = a_i' + a_i''$,

$$\nu(\mathcal{a}) \leq \max \{\nu(\mathcal{a}_{a_i - a_i'}), \nu(\mathcal{a}_{a_i - a_i''})\}.$$

ν is called K-valued m-norm on X and (X, ν) is called m-normed space over valued field K.

If valuation of K is non-archimedean, (X, ν) is called non-archimedean m-normed space.

Examples [16].

1. X is a vector space over R over which a norm is defined and F denote the collection of all bounded linear functionals f on X such that

$$\sup_{\nu(a) \leq 1} f(a) \leq 1. \ \mathcal{F} = (f_1, f_2, \ldots, f_m) \in F^{N_o}$$

$$D(\mathcal{F}, \mathcal{a}) = \begin{vmatrix} f_1(a_1), \ldots, f_1(a_m) \\ f_2(a_1), \ldots, f_2(a_m) \\ f_m(a_1), \ldots, f_m(a_m) \end{vmatrix} .$$

(X, ν) is a m-normed space where $\nu(\mathfrak{a}) = \nu(a_1, a_2, \ldots, a_m)$

$= \frac{1}{m!} \sup_{\gamma \in Y^{M_0}} D(\gamma, \mathfrak{a})$.

2. If X is a Hilbert space, then by Theorem 6 [18] we can define m-norm ν^*

$$\nu^*(a_1, a_2, \ldots, a_m) = \frac{1}{m!} \left\{ \begin{vmatrix} (a_1, a_1), \ldots, (a_1, a_m) \\ \vdots \\ (a_m, a_1), \ldots, (a_m, a_m) \end{vmatrix} \right\}^{1/2}$$

6.2. <u>FUNDAMENTAL PROPERTIES</u>

<u>Theorem</u> 6.2.1. A m-normed space (X, ν) is non-archimedean if and only if for each $\mathfrak{a} = (a_i)_{i \in M_0} \in X^{M_0}$ and points $a \in X$,

$$\nu(a_1, a_2, \ldots, a_m + a) \leq \max\{\nu(\mathfrak{a}), \nu(\mathfrak{a}_{a_m - a})\} \text{ holds.}$$

<u>Proof</u>.

1. Assume $\nu(a_1, a_2, \ldots, a_m + a) \leq \max\{\nu(\mathfrak{a}), \nu(\mathfrak{a}_{a_m - a})\}$ for any $\mathfrak{a} = (a_i)_{i \in M_0} \in X^{M_0}$ and $a \in X$ holds. Then the m-metric σ associated with m-norm ν satisfies the condition for $\mathfrak{a} = (a_i) \in X^{M_0}$ and $a \in X$,

$$\sigma(a_0,a_1,\dots,a_m) = \nu(a_1 - a_0,\dots,a_m-a_0)$$

$$= \nu((a_1-a) -(a-a_0),\dots,(a_{m-1}-a)-(a-a_0),$$
$$(a_m-a) - (a-a_0))$$

$$\leq \max\{\nu((a_1-a)-(a-a_0),\dots,(a_{m-1}-a)-$$
$$-(a-a_0),(a_m-a)),\ \nu((a_1-a)-(a-a_0),\dots$$
$$\dots(a_{m-1}-a)-(a-a_0),(a-a_0))\}$$

$$\leq \max\{\nu((a_1-a)-(a-a_0),\dots,(a_{m-1}-a),$$
$$(a_m-a)),$$
$$\nu((a_1-a)-(a-a_0),\dots,(a_m-a_0),$$
$$(a_m-a)),$$
$$\nu((a_1-a)-(a-a_0),\dots,(a_{m-1}-a_0),$$
$$(a-a_0))\}$$

$$\leq \max\{\sigma(-ma,a_2-a_1-ma,\dots,a_{m-1}-a_1-ma,$$
$$a_m+a-a_1-ma,0),$$

$$\sigma(-ma,a_2-a_1-ma,\dots,a_{m-1}-a_1-ma,0,a_1-ma),$$
$$\dots\dots\dots\dots\dots$$

$$\sigma(0, a_2-a_1-\alpha a, \ldots, a_{m-1}-a_1-\alpha a, a_m+a-a_1\alpha a, -a_1-\alpha a)\}$$

$$= \max\{\nu(-\alpha a, a_2-a_1-\alpha a, \ldots, a_{m-1}-a_1-\alpha a, a_m+a-a_1-\alpha a),$$

$$\nu(-\alpha a, a_2-a_1-\alpha a, \ldots, a_{m-1}-a_1-\alpha a, -a_1-\alpha a),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\nu(a_2-a_1-\alpha a, \ldots, a_{m-1}-a_1-\alpha a, a_m+a-a_1-\alpha a, -a_1-\alpha a)\}$$

$$= \max\{|\alpha|\ \nu(a, a_2-a_1, \ldots, a_{m-1}-a_1, a_m-a_1),$$

$$|\alpha|\ \nu(a, a_2-a_1, \ldots, a_{m-1}-a_1, a_1),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\nu(a_2, \ldots, a_{m-1}, a_m+a, a_1+\alpha a)\}.$$

Now,

$$\nu(a_2, \ldots, a_{m-1}, \alpha a_m+\alpha a, a_1+\alpha a)$$

$$= \nu(a_2-\alpha a+\alpha a, \ldots, a_{m-1}-\alpha a+\alpha a, \alpha a_m+\alpha a, a_1+\alpha a)$$

$$= \sigma(a_2-\alpha a, \ldots, a_{m-1}-\alpha a, \alpha a_m, a_1, -\alpha a)$$

$$\leq \max\{\sigma(a_2-\alpha a, \ldots, a_{m-1}-\alpha a, \alpha a_m, a_1, 0),$$

$$\sigma(a_2-\alpha a, \ldots, a_{m-1}-\alpha a, \alpha a_m, 0, -\alpha a)$$

$$\sigma(a_2-\alpha a, \ldots, a_{m-1}-\alpha a, 0, a_1, -\alpha a)$$

$$\cdots\cdots\cdots\cdots$$

$$\sigma(0, \ldots, a_{m-1}-\alpha a, \alpha a_m, a_1, -\alpha a)\}$$

$$= \max\{\nu(a_2-\alpha a, \ldots, a_{m-1}-\alpha a, \alpha a_m, a_1),$$

$$\nu(a_2-\alpha a, \ldots, a_{m-1}-\alpha a, \alpha a_m, -\alpha a),$$

$$\nu(a_2-\alpha a, \ldots, a_{m-1}-\alpha a, a_1, -\alpha a)$$

$$\cdots\cdots\cdots\cdots$$

$$\nu(a_3-\alpha a, \ldots, a_{m-1}-\alpha a, \alpha a_m, a_1, -\alpha a)\}$$

$$= \max\{\ |\alpha|\,\nu(a_2-\alpha a, \ldots, a_{m-1}-\alpha a, a_m, a_1),$$

$$|\alpha|^2\,\nu(a_2-\alpha a, \ldots, a_{m-1}-\alpha a, a_m, a),$$

$$|\alpha|\ \nu(a_2-\alpha a, \ldots, a_{m-1}-\alpha a, a_1, -a),$$

$$\cdots\cdots\cdots\cdots,$$

$$|\alpha|^3\nu(a_3-\alpha a, \ldots, a_{m-1}-\alpha a, a_m, a_1, a)\}$$

$$= \max\{|\alpha|\ \nu(a_2-\alpha a, \ldots, a_{m-1}-\alpha a, a_m, a_1),$$

$$|\alpha|^2\nu(a_2, \ldots, a_{m-1}, a_m, a),$$

$$|\alpha|\ \nu(a_2, \ldots, a_{m-1}, a_1, a),$$

$$\cdots\cdots\cdots$$

$$|\alpha|^2\nu(a_3, \ldots, a_{m-1}, a_m, a_1, a)\}.$$

Therefore

$$\nu(a_2,\dots,a_{m-1}, a_m+a, a_1+\alpha a)$$

$$\leq \max\{\nu(a_2-\alpha a,\dots, a_{m-1}-\alpha a, a_m, a_1),$$

$$|\alpha|\, \nu(a_2,\dots, a_{m-1}, a_m, a),$$

$$\nu(a_2,\dots, a_{m-1}, a_1, a),$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$|\alpha|\, \nu(a_3,\dots, a_{m-1}, a_m, a_1, a)\}.$$

For α different from 0 of K, therefore we have

$$\nu(a_1, a_2,\dots, a_{m-1}, a_m + a)$$

$$\leq \max\{\nu(a_2-\alpha a,\dots, a_{m-1}-\alpha a, a_m, a_1),$$

$$\nu(a_2,\dots, a_{m-1}, a_1, a),$$

$$|\alpha|\, \nu(a_2,\dots, a_{m-1}, a_m, a).$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$|\alpha|\, \nu(a_3,\dots, a_{m-1}, a_m, a_1, a),$$

$$|\alpha|\, \nu(a, a_2-a_1,\dots, a_{m-1}-a_1, a_m-a_1)$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$|\alpha|\, \nu(a, a_2-a_1,\dots, a_{m-1}-a_1, a_1)\}.$$

Since the valuation of K is non-trivial $a \to e$ is possible and so we get,

$$\nu(a_1, a_2, \ldots, a_{m-1}, a_m + a)$$

$$\leq \max \{ \nu(a_2, \ldots, a_{m-1}, a_m, a_1),$$

$$\nu(a_2, \ldots, a_{m-1}, a_1, a) \}$$

$$= \max \{ \nu(a_1, \ldots, a_m),$$

$$\nu(a_1, \ldots, a_{m-1}, a) \}$$

$$= \max \{ \nu(\mathfrak{a}), \nu(\mathfrak{a}_{a_m \to a}) \}.$$

This completes the proof of the theorem.

__Corollary__ 6.2.1. If (X, ν) is non-archimedean m-normed space then the valuation of K is also non-archimedean.

__Proof__. For any natural number n and the identity element e of K

$$|ne| \nu(\mathfrak{a}) = |ne| \nu(a_1, \ldots, a_m) = \nu(a_1, \ldots, n a_m)$$

$$\leq \max \{ \underbrace{\nu(\mathfrak{a}), \ldots, \nu(\mathfrak{a})}_{n \text{ times}} \} = \nu(\mathfrak{a}).$$

we get

$$| na | \leq 1 .$$

<u>Theorem</u> 6.2.2. A n-normed space (X, ν) is non-archimedean iff for any points $a_1, a_2, \ldots, a_n, a$ of X with $\nu(a_1,\ldots,a_n) < \nu(a_1,\ldots, a_{n-1},a)$ the relation $\nu(a_1,\ldots,a_n+a) = \nu(a_1,\ldots, a_{n-1},a)$ holds.

<u>Proof</u>. 1. Suppose that the condition given in the theorem is satisfied and let $a_1,\ldots, a_n, a$ be any points of X. If $\nu(a_1,\ldots, a_{n-1}, a) < \nu(a_1,\ldots, a_n + a)$ then

$$\nu(a_1,\ldots, a_{n-1},a) = \nu(a_1,\ldots, a_{n-1}, a_n + a-a_n),$$

$$= \nu(a_1,\ldots,a_{n-1},a_n+a).$$

If

$$\nu(a_1,\ldots, a_n) < \nu(a_1,\ldots, a_n + a) \quad \text{then}$$

$$\nu(a_1,\ldots, a_n) = \nu(a_1,\ldots, a_n+a-a)$$

$$= \nu(a_1,\ldots, a_n + a).$$

This implies that

$$\nu(a_1,\dots,a_m+a) \leq \max\{\nu(a_1,\dots,a_m), \nu(a_1,\dots,a_{m-1},a)\}$$

i.e. (X, ν) is non-archimedean.

2. If (X, ν) is non-archimedean so for any $a_1, a_2, \dots, a_m$, a of X with

$$\nu(a_1,\dots, a_m) < \nu(a_1,\dots, a_{m-1}, a)$$

the relation

$$\nu(a_1,\dots,a_m+a) \leq \max\{\nu(a_1,\dots,a_m), \nu(a_1,\dots,a_{m-1},a)\}$$

$$< \nu(a_1,\dots,a_{m-1},a)$$

$$\nu(a_1,\dots,a_{m-1},a) = \nu(a_1,\dots,a_{m-1}, a_m+a-a_m)$$

$$\leq \max\{\nu(a_1,\dots,a_m+a), \nu(a_1,\dots,a_{m-1},a_m)\}$$

$$\leq \nu(a_1,\dots,a_m+a) .$$

Thus $\nu(a_1,\dots, a_m+a) = \nu(a_1,\dots,a_{m-1}, a)$ holds.

This proves the theorem.

Theorem 6.2.3. Every non-archimedean n-normed space of dimension $\geq n$ is a locally K-convex topological vector space.

Proof. Slightly modefying the proof of Theorem 42 [18] we see that, the mappings $(a,b) \longrightarrow a-b$ of $X \times X$ into X and $(\alpha,a) \longrightarrow \alpha a$ of $(-\infty, \infty) \times X \longrightarrow X$ are continuous. We are further required to show that all sets $W_\Sigma(o)$, $\Sigma \in G$ are K-convex. Let $\alpha, \beta \in K$ and $|\alpha| \leq 1$, $|\beta| \leq 1$. For $a, b \in W_\Sigma(o)$ where

$$\Sigma = \{(\mathfrak{a}_1, \epsilon_1), \ldots, (\mathfrak{a}_n, \epsilon_n)\} \quad \mathfrak{a}_j = (a_{ji})_i \in M_{n1} \in X^{M_{n1}}$$

we have

$$\begin{aligned} \sigma[o, \alpha a + \beta b, \mathfrak{a}_j] &= \nu(\alpha a + \beta b, a_{j2} \ldots a_{jn}) \\ &\leq \max\{|\alpha| \nu(a, a_{j2} \ldots a_{jn}), |\beta| \nu(b, a_{j2} \ldots a_{jn})\} \\ &\leq \max\{\nu(a, a_{j2} \ldots a_{jn}), \nu(b, a_{j2} \ldots a_{jn})\} \\ &\leq \epsilon_j \quad \text{for } j = 1, \ldots, n. \end{aligned}$$

Thus $\alpha a + \beta b \in W_\Sigma(o)$.

This proves the theorem.

CHAPTER - VII

ON THE GENERALIZATION OF A THEOREM OF MONNA

In the present chapter, we prove an elegant theorem which is a generalisation of Monna's theorem concerning characterization of convergence preserving matrices. This is non-archimedean version of Robinson's theorem which contains well known Silverman-Toeplitz theorem as special case.

7.1. MONNA'S THEOREM

Let K be a complete non-archimedean valued field and $A = (a_{m,n})$ be an infinite matrix $a_{mn} \in K$, $m,n = o,1,2,..$ For each $m = o,1,2,....,$ let $t_m = \sum_{n=o}^{\infty} a_{m,n}\, s_n$, where $\{s_n\}$ is a sequence in a non-archimedean Banach space X over K, be convergent. The matrix A is called convergence preserving if $\{t_m\}$ is convergent whenever $\{s_n\}$ is convergent. It is called regular if limits of $\{t_m\}$ and $\{s_n\}$ are the same.

Monna [43] has obtained the following theorem.

Theorem 7.1.1. A necessary and sufficient conditions in order that $A = (a_{mn})$ be convergence preserving are :

(i) There exists constant $M > o$ such that

$$\sup_{m,n} | a_{m,n} | < M$$

(ii) $\lim_{m\to\infty} a_{m,n} = a_n$, $n = o,1,2,\ldots$

(iii) $\lim_{m\to\infty} \sum_{n=o}^{\infty} a_{m,n} = a.$

It is easy to see that A is regular if, and only if

$a_n = o$, $n = o,1,2,\ldots$

$a_1 = 1.$

7.2. GENERALIZATION OF MONNA'S THEOREM

Let $\{A_{k,n}\}$ be a double sequence of bounded linear operators of non-archimedean Banach space X into itself. For $\{s_n\} \in X$ define

$$t_k = \sum_{n=1}^{\infty} A_{k,n}\, s_n \in X.$$

Theorem 7.2.1. The necessary and sufficient conditions that $\{t_k\}$ is convergent whenever $\{s_n\}$ is convergent are :

(i) $\lim_k A_{k,n} = o$ for all n

(ii) $\lim_{n\to\infty} A_{k,n} = o$[1] for all k

(Equivalently $\sum_n A_{k,n} = A_k$ exists)

(iii) $\lim_k A_k = I$[2].

---

1). O is zero operator

2). Identity operator

(iv) $\sup_{n,k} \| A_{k,n} \| = M < \infty$.

In the proof of this theorem we need the following lemma.

Lemma 7.2.1. Let $\{T_p\}$ be a sequence of non-archimedean Banach space X over valued field K. Then $a_n \longrightarrow a$ implies $T_p\, a_p \longrightarrow o$ ($\sum_p T_p\, a_n < \infty$) if, and only if $T_p \longrightarrow o$ pointwise.

Proof of Lemma 7.2.1. Sufficiency : Suppose $a_n \longrightarrow o$ implies $T_p \longrightarrow o$ pointwise then we have to show that $T_p \longrightarrow o$ pointwise.

By the hypothesis $\sup \| T_p \| = M < \infty$ in view of Banach Steinhaus theorem.

For $\epsilon > o$ there eixists p_o such that $\| a_p \| < \epsilon / M$ for $p \geq p_o$. $\|T_p\, a_p\| = \| T_p a_p \dfrac{\| a_p \|}{\|a_p\|} \| = \|a_p\|\, \|T_p a'_p\|$

where $a'_p = \dfrac{a_p}{\|a_p\|}$, $\|a'_p\| = 1$.

or $\| T_p a_p \| \leq \|a_p\| \sup \|T_p a\| < \epsilon/_M M = \epsilon$, that is, $T_p \longrightarrow o$ pointwise.

Necessity : Suppose $T_p \longrightarrow o$ pointwise and $s_p \longrightarrow s$. Then

$$||T_p s_p|| \leq \max \left[||T_p s||, ||T_p(s_p - s)|| \right] \to o$$

as $p \to \infty$.

Hence $T_p s_n \longrightarrow o$ as $p \longrightarrow \infty$.

This proves the lemma.

Proof of Theorem 7.2.1.

Sufficiency : Fix $\epsilon > o$ then we are required to prove that $||t_k - s|| < \epsilon$ for k sufficiently large. We have

$$|| s_n - s || < \epsilon / M , \; n \geq N$$

and

$$|| t_k - s || = || \sum_{n=1}^{\infty} A_{k,n} s_n - I_s ||$$

$$= || \sum_{n=1}^{\infty} A_{k,n}(s_n - s) + \sum_{n=1}^{\infty} A_{k,n} s - I_s ||$$

$$\leq \max \Big\{ || \sum_{n=1}^{\infty} A_{k,n}(s_n - s) ||,$$

$$|| \sum_{n=N+1}^{\infty} A_{k,n} (s_n - s) ||, || \sum_{n=1}^{\infty} (A_{k,n} - I)s || \Big\}.$$

Further

$$||\sum_{n=1}^{\infty} A_{k,n}(s_n-s)|| = \max_{n=1,2,3,\ldots N} ||A_{k,n}(s_n-s)||$$

$$\leq \epsilon \text{ for } k > K$$

$$A_{k,1}(s_1-s) \longrightarrow o \quad k > k_1$$

$$A_{k,2}(s_2-s) \longrightarrow o \quad k > k_2$$

$$||\sum_{n=N+1}^{\infty} A_{k,n}(s_n-s)|| \leq \max ||A_{k,n}(s_n-s)||$$

$$\leq \max (||A_{k,n}|| \; ||s_n-s||)$$

$$\leq M \;\; \epsilon/M \; < \; \epsilon \;\; \text{for all } K.$$

$$||\sum_{n=1}^{\infty} (A_{k,n}-I)s|| \leq \max_n ||(A_{k,n}-I)s||$$

$$\leq \epsilon \text{ for } k > T$$

Therefore $||t_k-s|| < \epsilon$ for $k > K, T$.

<u>Necessity</u> :

(i) Take $x \in X$, choose $A_n = (o,o,\ldots,o, \; ,o,\ldots) \to o$

Then $t_k = \sum_{n=1}^{\infty} A_{k,n} s_n = A_{k,n} x \to o$ for all x.

which implies that $\lim_{k} A_{k,n} = o$.

(ii) Let $s_n \to o$ then $t_k = \sum_{n=1}^{\infty} A_{k,n} s_n \to o$ for all k.

By the Lemma 7.2.1 $A_{k,n} \to o$ as $n \to \infty$ for all k pointwise.

(iii) For $x \in X$, $\lim A_k = x$

$s_n = (x, x, \dots) \to x$ then

$t_k = \sum_{n=1}^{\infty} A_{k,n} x = (\sum_{n=1}^{\infty} A_{k,n}) x = A_k x \to x$. Thus

$$\lim_{k} A_k = I.$$

(iv) Suppose there exists k_1 such that

$\sup_{n} \| A_{k_1,n} x \| = \infty$. Then in view of the Lemma 7.2.1. there exists $\{s_n\}$ belong to X convergent such that $A_{k_1,n} s_n$ does not convergent to o. This implies that $t_{k_1} = \sum_{n=1}^{\infty} A_{k_1,n} s_n$ divergent which is a contradiction. Therefore for all k, $\sup_{n} \| A_{k,n} \| = M_k < \infty$.

Let $T_k : c_o(X)$[1] $\longrightarrow X$, $\sigma = (s_n) \longrightarrow \sum_{n=1}^{\infty} A_{kn} s_n$

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1). $c_o(X)$ denote the nonarchimedean Banach space of convergent sequences with limit o in X.

$T_k(\sigma) \longrightarrow e$ for all $\sigma \in c_0(X)$ or $t_k \longrightarrow e$ pointwise which implies that $(\|T_k\|)_k$ bounded by Banach-Steinhaus theorem.

Now we prove that $\| T_k \| = M_k$.

$$\|T_k(\sigma)\| = \Big\|\sum_{n=1}^{\infty} A_{n,k} a_n\Big\| \leq \max_n \|A_{nk} a_n\|$$

$$\leq M_k \max_n \|a_n\|$$

$$\leq M_k \|\sigma\|$$

This implies $\| T_k \| \leq M_k$ ...(7.2.1)

Take $\tau = (0, 0, \ldots, x, 0, \ldots)$ with $\| x \| \leq 1$, then

$$\|A_{nk} x\| = \| T_k \tau\| \leq \| T_k \| \, \|\tau\|$$

$$\leq \| T_k \|$$

This implies that

$$\| A_{n,k} \| \leq \| T_k \|$$

or $\sup \| A_{n,k} \| \leq \| T_k \|$

or $M_k \leq \| T_k \|$ ...(7.2.2)

By (7.2.1) and (7.2.2)

$$M_k = \| x_k \| .$$

Thus we have proved that

$$\sup_{n,k} \| A_{k,n} \| = M < \infty .$$

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